Math 240: Linear Differential Equations

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Outline

Review for Last Time

1. Construct general solutions to homogeneous and nonhomogeneous linear D.E.s
2. Use auxiliary equations to solve constant coefficient linear homogeneous D.E.s

Auxiliary Equations

Given a linear homogeneous **constant-coefficient** differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... a_1 \frac{dy}{dx} + a_0 y = 0,$$

the **Auxiliary Equation** is

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The Auxiliary Equation determines the general solution.

General Solutions to Nonhomogeneous Linear D.E.s

Theorem

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation on an interval I. Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions to the associated homogeneous differential equation. Then the general solution to the nonhomogeneous equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p$$

where the c_i are arbitrary constants.

General Solution from the Auxiliary Equation

1. If m is a root of the auxiliary equation of multiplicity k then e^{mx}, xe^{mx}, x^2e^{mx}, ... , $x^{k-1}e^{mx}$ are linearly independent solutions.
2. If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are a roots of the auxiliary equation of multiplicity k then $e^{\alpha x}cos(\beta x)$, $xe^{\alpha x}cos(\beta x)$, ... , $x^{k-1}e^{\alpha x}cos(\beta x)$ and $e^{\alpha x}sin(\beta x)$, $xe^{\alpha x}sin(\beta x)$, ... , $x^{k-1}e^{\alpha x}sin(\beta x)$ are linearly independent solutions.

Today's Goals

1. Learn how to solve nonhomogeneous linear differential equations using the method of Undetermined Coefficients.

The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... a_1 y' + a_0 y = g(x)$$

where $a_n, a_{n-1}, ..., a_0$ are constants.

The Method of Undetermined Coefficients

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2. Step 2: Find a particular solution by analyzing g(x) and making an educated guess.
3. Step 3: Add the homogeneous solution and the particular solution together to get the general solution.

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e^{4x}	Ae^{4x}
x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
$e^{2x}cos(4x)$	$Ae^{2x}sin(4x) + Be^{2x}cos(4x)$
$3xsin(5x)$	$(Ax + B)sin(5x) + (Cx + D)cos(5x)$
$xe^{2x}cos(3x)$	$(Ax + B)e^{2x}sin(3x) + (Cx + D)e^{2x}cos(3x)$

The Guessing Rule

The form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiation of $g(x)$.

A Problem

Solve $y'' - 5y' + 4y = 8e^x$ using undetermined coefficients.

The solution

When the natural guess for a particular solution duplicates a homogeneous solution, multiply the guess by x^n, where n is the smallest positive integer that eliminates the duplication.