Math 240: Linear Differential Equations

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Outline

Review for Last Time

1. Higher order linear differential equations.
2. Superposition principal for higher order linear homogeneous differential equations.
3. Testing for linear independence of functions.

1. The following is a general nth-order linear D.E.

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

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2. For a linear homogeneous D.E., linear combinations of solutions are again solutions.
3. A collection of functions is linearly independent if and only if the Wronskian is non-zero

Today's Goals

1. Construct general solutions to homogeneous and nonhomogeneous linear D.E.s
2. Use auxiliary equations to solve constant coefficient linear homogeneous D.E.s

Linearly Independent Solutions

Theorem

Let $y_1, y_2, ..., y_n$ be n solutions to a homogeneous linear nth-order differential equation on an interval I. The the set of solutions is **linearly independent** *on I if and only if $W(y_1, y_2, ..., y_n) \neq 0$ for every x in the interval.*

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Note: There always exists a fundamental set of solutions to an nth-order linear homogeneous differential equation on an interval I.

General Solution

Theorem

Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions set of solutions to an nth-order linear homogeneous differential equation on an interval I. Then the general solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

where the c_i are arbitrary constants.

General Solutions to Nonhomogeneous Linear D.E.s

Theorem

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation on an interval I. Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions to the associated homogeneous differential equation. Then the general solution to the nonhomogeneous equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p$$

where the c_i are arbitrary constants.

Superposition Principle for Nonhomogeneous Equations

Theorem

Suppose y_{p_i} denotes a particular solution to the differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

Where $i = 1, 2, ..., k$. Then $y_p = y_{p_1} + y_{p_2} + ... + y_{p_k}$ is a particular solution of

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y =$$

$$g_1(x) + g_2(x) + ... + g_k(x)$$

A Motivating Example

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In this case, we get $e^{mx}(am^2 + bm + c) = 0$. There are three possibilities for the roots of a quadratic equation.

Case 1: Distinct Roots

If $am^2 + bm + c$ has distinct roots m_1 and m_2, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

.

Case 2: Repeated Roots

If $am^2 + bm + c$ has a repeated root m_1, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

.

Case 3: Complex Roots

If $am^2 + bm + c$ has complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha x} cos(\beta x) + ic_2 e^{\alpha x} sin(\beta x)$$

.

Auxiliary Equations

Given a linear homogeneous **constant-coefficient** differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1 \frac{dy}{dx} + a_0 y = 0,$$

the **Auxiliary Equation** is

$$a_n m^n + a_{n-1} m^{n-1} + \ldots a_1 m + a_0 = 0.$$

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the **Auxiliary Equation** is

$$a_n m^n + a_{n-1} m^{n-1} + \ldots a_1 m + a_0 = 0.$$

The Auxiliary Equation determines the general solution.

General Solution from the Auxiliary Equation

1. If m is a root of the auxiliary equation of multiplicity k then e^{mx}, xe^{mx}, x^2e^{mx}, ... , $x^{k-1}e^{mx}$ are linearly independent solutions.
2. If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are a roots of the auxiliary equation of multiplicity k then $e^{\alpha x}cos(\beta x)$, $xe^{\alpha x}cos(\beta x)$, ... , $x^{k-1}e^{\alpha x}cos(\beta x)$ and $e^{\alpha x}sin(\beta x)$, $xe^{\alpha x}sin(\beta x)$, ... , $x^{k-1}e^{\alpha x}sin(\beta x)$ are linearly independent solutions.