Math 240: Eigenvalues

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Outline

Review of last time

1. How to find the inverse of a Matrix.
2. Properties of inverses.
3. How to use inverses to solve a linear system.

Inverses of Arbitrary $n \times n$ Matrices

How to find the inverse of an arbitrary $n \times n$ matrix A.

1. Form the augmented $n \times 2n$ matrix $[A|I_n]$.
2. Find the reduced row echelon form of $[A|I_n]$.
3. If $rank(A) < n$ then A is not invertible.
4. If $rank(A) = n$, then the RREF form of the augmented matrix is $[I_n|A^{-1}]$.

Solving a Linear System Using Inverses

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Example: Solve the following linear system using inverses.

$$x + y = 4$$

$$2x - y = 14$$

Today's Goals

1. Know how to interpret matrices as maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.
2. Know how to find eigenvalues.
3. Know how to find eigenvectors.

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

The following are examples of functions from $\mathbb{R}^n$ into $\mathbb{R}$.

1. $f(x_1) = 2x_1$
2. $f(x_1) = x_1^2$
3. $f(x_1, x_2) = x_1^2 + x_2^2$
4. $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i^2$

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The following are examples of extending the above functions to functions from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$.

1. $g(x_1) = (x_1, 2x_1)$
2. $g(x_1) = (x_1, {x_1}^2)$
3. $g(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$
4. $g(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, \sum_{i=1}^{n} x_i^2)$

General maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

The following is a general map from $\mathbb{R}^n$ with coordinates $x_1, x_2, ..., x_n$ to $\mathbb{R}^m$ with coordinates $w_1, w_2, ..., w_m$.

$w_1 = f_1(x_1, x_2, ..., x_n)$

$w_2 = f_2(x_1, x_2, ..., x_n)$

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$w_m = f_m(x_1, x_2, ..., x_n)$

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Example: Rewrite $g(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$ in this form.

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$w_1 = a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n$

$w_2 = a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n$

$\ldots$

$w_m = a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n$

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Key idea: Matrix-vector multiplication always encodes a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ and every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be encoded as Matrix-vector multiplication.

Types of Linear Maps

The following are types of linear maps

1. Reflection about a line in R^2
2. Reflection about a plane in R^3
3. Orthogonal projection onto an axis in R^2
4. Orthogonal projection onto a plane in R^3
5. Rotation about the origin in R^2

Eigenvalue and Eigenvector

Definition

Let λ be a scalar, x be a $n \times 1$ column vector and A be a $n \times n$ matrix. Any nontrivial vector that solves $Ax = \lambda x$ is called an **eigenvector**. If $Ax = \lambda x$ has a non-trivial solution, λ is an **eigenvalue**.

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Key idea: Eigenvectors are vectors sent to scalar copies of themselves under the linear map corresponding to A.

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To find eigenvalues we want to solve $Ax = \lambda x$ for λ.

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$Ax - \lambda x = 0$

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Hence, to find the eigenvalues, we solve the polynomial equation $det(A - \lambda I_n) = 0$ called the **characteristic equation**.

Finding Eigenvectors

For each eigenvalue λ, solve the linear system $(A - \lambda I_n)x = 0$ to find the eigenvectors.