# Math 240: Inverses

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# Outline

# Review of last time

1. How to find determinants using cofactor expansion.
2. How to find determinants using row operations.
3. Properties of determinants.

# Definition of Arbitrary Determinant

## Definition

Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix.
The cofactor expansion of $A$ along the ith row is

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

The cofactor expansion of $A$ along the jth column is

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

# Using Elementary Row Operations to Find the Determinant

Suppose $B$ is obtained from $A$ by:

1. multiplying a row by a non-zero scalar $c$, then $det(A) = \frac{1}{c}det(B)$.
2. switching rows, then $det(A) = -det(B)$.
3. adding a multiple of one row to another row, then $det(A) = det(B)$.

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Using this idea we can quickly find determinants by row-reducing to triangular form.

# Properties of Determinants

## Theorem

*If elementary row or column operations lead to one of the following conditions, then the determinant is zero.*

1. *an entire row (or column) consists of zeros.*
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Let $A$ and $B$ be $n \times n$ matrices and $c$ be a scalar.

1. $det(AB) = det(A)det(B)$
2. $det(cA) = c^n det(A)$
3. $det(A^T) = det(A)$

# Today's Goals

1. Be able to find the inverse of a matrix or show it has no inverse.
2. Know the properties of inverses.
3. Be able to solve systems of linear equations using matrices.

# Matrix Inverse

## Definition

An $n \times n$ matrix $A$ is **invertible** if there exists an $n \times n$ matrix $B$ such that

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2. A matrix that is not invertible is called **singular**.
3. If $A$ is invertible, its inverse is denoted $A^{-1}$.

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**Example:** check the the following matrices are inverses of each other.

$$\begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} -7 & 2 \\ 4 & -1 \end{pmatrix}$$

# A $2 \times 2$ Matrix Inverse Formula

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $2 \times 2$ matrix and $det(A) \neq 0$, then

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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**Exercise:** Prove the above statement

# Inverses of Arbitrary $n \times n$ Matrices

How to find the inverse of an arbitrary $n \times n$ matrix $A$.

1. Form the augmented $n \times 2n$ matrix $[A|I_n]$.
2. Find the reduced row echelon form of $[A|I_n]$.
3. If $rank(A) < n$ then $A$ is not invertible.
4. If $rank(A) = n$, then the RREF form of the augmented matrix is $[I_n|A^{-1}]$.

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Find the inverse of $\begin{pmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

# Properties of Inverses

1. $(A^{-1})^{-1} = A$
2. $(cA)^{-1} = \frac{1}{c}A^{-1}$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$
5. $det(A^{-1}) = \frac{1}{det(A)}$
6. $A$ is invertible if and only if $det(A) \neq 0$

# Solving a Linear System Using Inverses

Let $A$ be invertible and $Ax = B$ be a linear system, then the solution to the linear system is given by

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**Example:** Solve the following linear system using inverses.

$$x + z = -4$$

$$x + y + z = 0$$

$$5x - y = 6$$