Math 240: Determinants

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Outline

Review of last time

1. Linear Independence
2. How to use row echelon form to determine linear independence
3. Rank of a matrix
4. How to use rank to determine consistency of a linear system

Linear independence and Rank

Definition

(Pragmatic)
Let A be an $m \times n$ matrix and B be its row-echelon form. The **rank** of A is the number of pivots of B.

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How to find if m vectors are linearly independent:

1. Make the vectors the rows of a $m \times n$ matrix (where the vectors are of size n)
2. Find the rank of the matrix.
3. If the rank is m then the vectors are linearly independent. If the rank is less than m, then the vectors are linearly dependant.

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Example: Are the following vectors linearly independent?

$$< 2, 6, 0, 1, -3 >, < -4, -12, 1, 1, 2 >, < 2, 6, 0, 1, 1 >$$

Determining Consistency

Given the linear system $Ax = B$ and the augmented matrix $(A|B)$.

1. If $rank(A) = rank(A|B) =$ the number of rows in x, then the system has a unique solution.
2. If $rank(A) = rank(A|B) <$ the number of rows in x, then the system has ∞-many solutions.
3. If $rank(A) < rank(A|B)$, then the system is inconsistent.

Today's Goals

1. Be able to find determinants using cofactor expansion.
2. Be able to find determinants using row operations.
3. Know the properties of determinants.

Determinant of a 2×2 Matrix

Definition

Give a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A is

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Minors and Cofactors

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Given a matrix A the **cofactor**, C_{ij}, is given by the following formula

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Definition of Arbitrary Determinant

Definition

Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix.
The cofactor expansion of A along the ith row is

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

The cofactor expansion of A along the jth column is

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Special Matrices and Determinants

Definition

An $n \times n$ matrix $A = (a_{ij})_{n\times n}$ is **lower triangular** if $a_{ij} = 0$ whenever $i < j$. An $n \times n$ matrix $A = (a_{ij})_{n\times n}$ is **upper triangular** if $a_{ij} = 0$ whenever $i > j$.

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If A is upper triangular, lower triangular or diagonal, then $det(A)$ is equal to the product of the diagonal entries.

Using Elementary Row Operations to Find the Determinant

Suppose B is obtained from A by:

1. multiplying a row by a non-zero scalar c, then $det(A) = \frac{1}{c}det(B)$.
2. switching rows, then $det(A) = -det(B)$.
3. adding a multiple of one row to another row, then $det(A) = det(B)$.

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Example: Find the determinant of the following matrix.

$$\begin{pmatrix} 0 & 2 & 0 & -3 \\ 3 & 0 & 2 & 5 \\ -2 & 4 & 0 & 6 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Properties of Determinants

Theorem

If elementary row or column operations lead to one of the following conditions, then the determinant is zero.

1. *an entire row (or column) consists of zeros.*
2. *one row (or column) is a multiple of another row (or column).*

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Let A and B be $n \times n$ matrices and c be a scalar.

1. $det(AB) = det(A)det(B)$
2. $det(cA) = c^n det(A)$
3. $det(A^T) = det(A)$