Math 240: Linear Systems and Rank of a Matrix

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Outline

Review of last time

1. Transpose of a matrix
2. Special types of matrices
3. Matrix properties
4. Row-echelon and reduced row echelon form
5. Solving linear systems using Gaussian and Gauss-Jordan elimination

Echelon Forms

Definition

A matrix is in **row-echelon form** if

1. Any row consisting of all zeros is at the bottom of the matrix.
2. For all non-zero rows the leading entry must be a one. This is called the **pivot**.
3. In consecutive rows the pivot in the lower row appears to the right of the pivot in the higher row.

Definition

A matrix is in **reduced row-echelon form** if it is in row-echelon form and every pivot is the only non-zero entry in its column.

Row Operations

We will be applying row operations to augmented matrices to find solutions to linear equations. This is called **Gaussian** or **Gauss-Jordan** elimination.

Here are the row operations:

1. Multiply a row by a number.
2. Switch rows.
3. Add a multiple of one row to another.

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Key Fact: If you alter an augmented matrix by row operations you preserve the set of solutions to the linear system.

Today's Goals

1. Be able to use rank of a matrix to determine if vectors are linearly independent.
2. Be able to use rank of an augmented matrix to determine consistency or inconsistency of a system.

Linear Independence

Definition

Let $v_1, ..., v_m$ be vectors in $\mathbb{R}^n$. The set $S = \{v_1, ..., v_m\}$ is **linearly independent** if $c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$ implies $c_1 = c_2 = ... = c_n = 0$.

If there exists a non trivial solution to $c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$ we say the set S is linearly dependant.

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Example: Are the following vectors linearly independent?

$$< 1, 2, 1 >, < 1, 1, 0 >, < 1, 0, 1 >$$

Definition

Let A be an $m \times n$ matrix. The **rank** of A is the maximal number of linearly independent row vectors

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(Pragmatic)
Let A be an $m \times n$ matrix and B be its row-echelon form. The **rank** of A is the number of pivots of B.

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Example What is the rank of the following matrix.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & -1 & -2 \end{pmatrix}$$

Determining Linear independence Using Matrices

How to find if m vectors are linearly independent:

1. Make the vectors the rows of a $m \times n$ matrix (where the vectors are of size n)
2. Find the rank of the matrix.
3. If the rank is m then the vectors are linearly independent. If the rank is less than m, then the vectors are linearly dependant.

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Example: Are the following vectors linearly independent?

$$< -2, 0, 4, 1 >, < 0, 0, 1, -1 >, < 0, 1, 0, 1 >, < 3, 2, -3, 0 >$$

Determining Consistency

Given the linear system $Ax = B$ and the augmented matrix $(A|B)$.

1. If $rank(A) = rank(A|B) =$ the number of rows in x, then the system has a unique solution.
2. If $rank(A) = rank(A|B) <$ the number of rows in x, then the system has ∞-many solutions.
3. If $rank(A) < rank(A|B)$, then the system is inconsistent.