MATH 104 HOMEWORK PRACTICE MIDTERM 2

NAME (PRINTED):

TA:

RECITATION TIME:

Please *turn off all electronic devices*. You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**, even on multiple choice or short answer questions—the grading will be based on your work shown as well as the end result. Please **clearly mark** a multiple choice option for each problem. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

Your signature

Problem	**Score (out of)**
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
6	(10)
7	(10)
8	(10)
Total	(80)

1. (10 pts) Find the length of the curve $y = \int_0^x \sqrt{cos(2t)}dt$ from $x = 0$ to $x = \frac{\pi}{4}$.

$$f(x) = \int_0^x \sqrt{\cos(2t)}\, dt$$

$$f'(x) = \frac{d}{dx}\left(\int_0^x \sqrt{\cos(2t)}\, dt \right) = \sqrt{\cos(2x)}$$

Arc length $= \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\left(\sqrt{\cos(2x)}\right)^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos(2x)}\, dx$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2(x)}\, dx$$

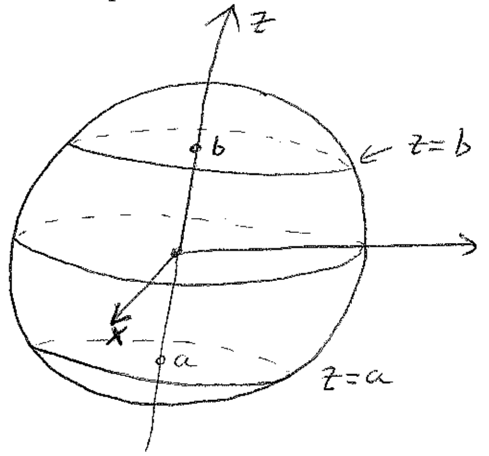
$$= \sqrt{2} \int_0^{\frac{\pi}{4}} \cos(x)\, dx$$

$$= \sqrt{2}\ \sin(x) \Big|_0^{\pi/4}$$

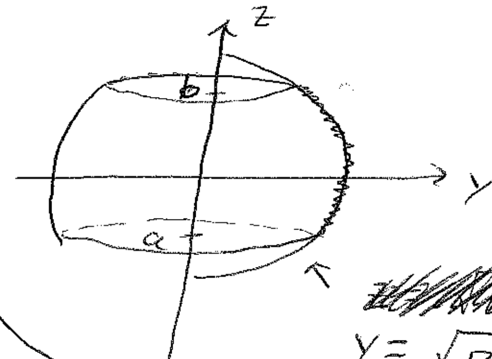
$$= \sqrt{2}\left(\frac{\sqrt{2}}{2} - 0 \right)$$

$$= \boxed{1}$$

2. (10 pts) Find the surface area of the portion of the sphere of radius R that lies between the planes $z = a$ and $z = b$ where $-R < a < b < R$.



we get this shape by rotating the curve $y=\sqrt{R^2-z^2}$ abot the z axis i.e.



$y=\sqrt{R^2-z^2}$

$$\frac{dy}{dz} = \frac{1}{2}(R^2-z^2)^{-1/2}(-2z) = \frac{-z}{\sqrt{R^2-z^2}}$$

$$SA = \int_a^b 2\pi\left(\sqrt{R^2-z^2}\right)\sqrt{1+\left(\frac{-z}{\sqrt{R^2-z^2}}\right)^2}\,dz$$

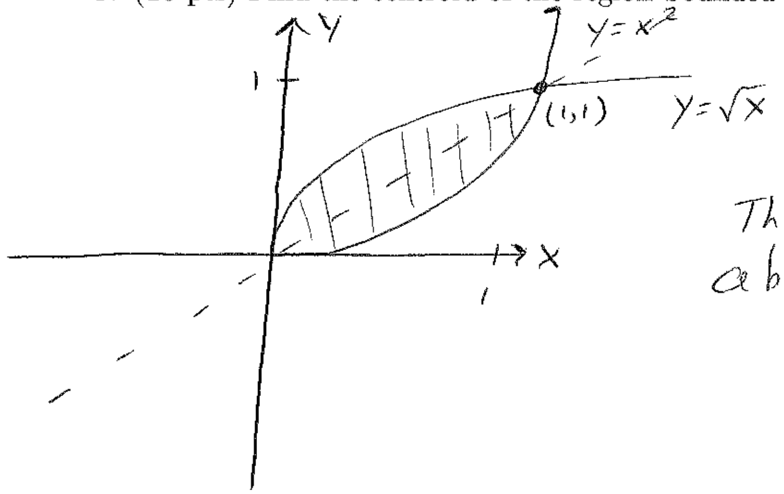
$$= \int_a^b 2\pi\sqrt{R^2-z^2}\sqrt{1+\frac{z^2}{R^2-z^2}}\,dz$$

$$= \int_a^b 2\pi\sqrt{R^2-z^2}\sqrt{\frac{R^2}{R^2-z^2}}$$

$$= \int_a^b 2\pi R$$

$$= \boxed{2\pi R(b-a)}$$

3. (10 pts) Find the centroid of the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$.



This region is symetric about the lin $y = x$

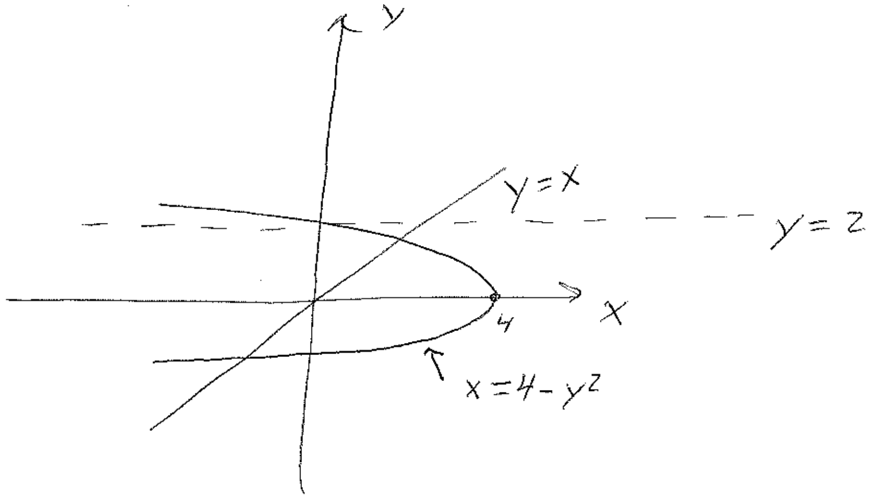
$$\overline{X} = \frac{\int_a^b x(f(x) - g(x))\,dx}{\int_a^b f(x) - g(x)\,dx} = \frac{\int_0^1 x(\sqrt{x} - x^2)\,dx}{\int_0^1 \sqrt{x} - x^2\,dx}$$

$$= \frac{\frac{2}{5}x^{5/2} - \frac{x^4}{4}\Big|_0^1}{\frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big|_0^1} = \frac{\frac{2}{5} - \frac{1}{4}}{\frac{2}{3} - \frac{1}{3}} = \frac{9}{20}$$

By symetry of the region about the line $y = x$,

$$\overline{Y} = \frac{9}{20}$$

4. (10 pts) Find the volume of the solid obtained by rotating the region bounded by $x = 4 - y^2$ and $y = x$ about the line $y = 2$.



Find pts of intersection

$$y = 4 - y^2$$

$$y^2 + y - 4 = 0$$

$$y = \frac{-1 \pm \sqrt{1 - 4(-4)}}{2}$$

$$y = \frac{-1 \pm \sqrt{17}}{2}$$ Oops... not so nice

Use shells method:

$$Vol = \int_a^b 2\pi (\text{radius})(\text{height})\, dy$$

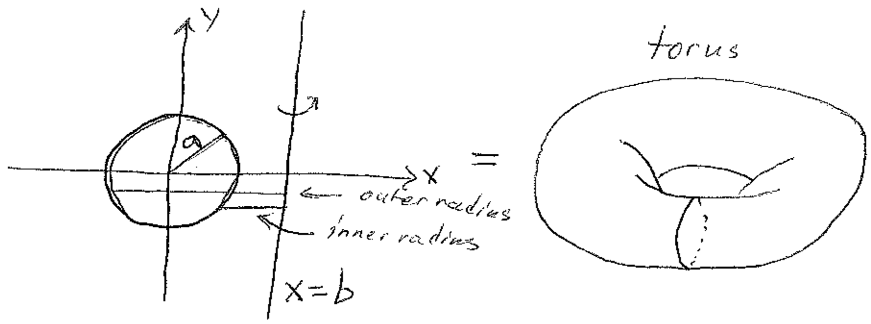
$$= \int_{\frac{-1-\sqrt{17}}{2}}^{\frac{-1+\sqrt{17}}{2}} 2\pi (2-y)\left((4-y^2) - y\right) dy$$

$$= \int_{\text{Stuff}}^{\text{stuff}} 2\pi (2-y)(-y^2 - y + 4)\, dy$$

$\vdots$ wolfram says

$$= \frac{85\sqrt{17}}{6}\pi$$

5. The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ (where $b > a$) to generate a solid shaped like a doughnut called a torus. Find its volume.



Use washer method

Inner radius $= b - \sqrt{a^2-y^2}$

Outer radius $= b + \sqrt{a^2-y^2}$

$$\text{Vol} = \int_{-a}^{a} \pi\left(b+\sqrt{a^2-y^2}\right)^2 - \pi\left(b-\sqrt{a^2-y^2}\right)^2 \, dy$$

$$= \int_{-a}^{a} \pi\left(b^2 + 2b\sqrt{a^2-y^2} + a^2 - y^2\right) - \pi\left(b^2 - 2b\sqrt{a^2-y^2} + a^2 - y^2\right) dy$$

$$= \pi \int_{-a}^{a} 4b\sqrt{a^2-y^2}\, dy$$

$$= 4\pi b \int_{-a}^{a} \sqrt{a^2-y^2}\, dy = 4\pi b \left(\text{area of half disk of radius } a\right)$$

$$= 4\pi b \left(\frac{\pi a^2}{2}\right)$$

$$= 2\pi^2 a^2 b$$

6. (10 pts) Find the following integrals or show they do not exist.

$$\int_{-1}^{2} x^{-\frac{3}{2}} dx$$

$$\int_{-1}^{2} x^{-\frac{2}{3}} dx$$

Note: Both $x^{-3/2}$ and $x^{-2/3}$ have discontinuities at $x=0$.

$$\int_{-1}^{2} x^{-3/2} dx = \lim_{a \to 0^-} \int_{-1}^{a} x^{-3/2} dx + \lim_{b \to 0^+} \int_{b}^{2} x^{-3/2} dx$$

$$= \lim_{a \to 0^-} -2x^{-1/2} \Big|_{-1}^{a} + \lim_{b \to 0^+} -2x^{-1/2} \Big|_{b}^{2}$$

$$= \underbrace{\lim_{a \to 0^-} -2a^{-1/2} - 2(-1)^{-1/2}}_{\text{does not converge}} + \underbrace{\lim_{b \to 0^+} -2(2)^{-1/2} + 2b^{-1/2}}_{\text{does not converge}}$$

$$\boxed{\int_{-1}^{2} x^{-3/2} dx \text{ does not converge}}$$

$$\int_{-1}^{2} x^{-2/3} dx = \lim_{a \to 0^-} \int_{-1}^{a} x^{-2/3} dx + \lim_{b \to 0^+} \int_{b}^{2} x^{-2/3} dx$$

$$= \lim_{a \to 0^-} 3x^{1/3} \Big|_{-1}^{a} + \lim_{b \to 0^+} 3x^{1/3} \Big|_{b}^{2}$$

$$= \lim_{a \to 0^-} 3a^{1/3} - 3(-1)^{1/3} + \lim_{b \to 0^+} 3(2)^{1/3} - 3(b)^{1/3}$$

$$= \boxed{-3\sqrt[3]{-1} + 3(2)^{1/3}}$$

7. (10 pts) Show that the following integral converges or show that it diverges.

$$\int_{e^e}^{\infty} ln(ln(x))dx$$

Let $f(x) = \ln(x)$

$f'(x) = 1/x$

Hence, $\ln(x)$ is increasing for $x > 0$.

This implies $\ln(\ln(x))$ is increasing for $x \geq 1$.

Since $\ln(\ln(e^e)) = \ln(e) = 1$, and $\ln(\ln(x))$ is increasing for $x \geq 1$, then $\ln(\ln(x)) \geq 1$ for $x \geq e^e$.

Hence $$\int_{e^e}^{\infty} 1\, dx \leq \int_{e^e}^{\infty} \ln(\ln(x))\, dx$$

$$\int_{e^e}^{\infty} 1\, dx = \lim_{a \to \infty} \int_{e^e}^{a} 1\, dx = \lim_{a \to \infty} x \Big|_{e^e}^{a}$$

$$= \lim_{a \to \infty} a - e^e$$

$$= \infty$$

Since $\int_{e^e}^{\infty} 1 dx$ diverges, then $\int_{e^e}^{\infty} \ln(\ln(x)) dx$ diverges.

8. (10 pts) Show that $f(x) = \frac{1}{2}e^{-|x|}$ is a probability density function

① Since $|X|$ and e^{-X} are continuous functions and composition of continuous functions is continuous, then $f(X)$ is continuous.

② Show $\frac{1}{2} e^{-|X|} \geq 0$ for $X \in \mathbb{R}$.

Since e^{X} is non-negative for any $X \in \mathbb{R}$, then

$\frac{1}{2} e^{-|X|} \geq 0$ for all $X \in \mathbb{R}$.

Since $f(x)$ is even.

③ Compute

$$\int_{-\infty}^{\infty} \frac{1}{2} e^{-|X|}\,dX = 2\int_0^{\infty} \frac{1}{2} e^{-X}\,dX$$

$$= \lim_{a\to\infty} \int_0^a e^{-X}\,dX$$

$$= \lim_{a\to\infty} -e^{-X}\Big|_0^a$$

$$= \lim_{a\to\infty} -e^{-a} + e^{-0}$$

$$= 0 + 1 = 1. \checkmark$$

By ①, ② and ③ $f(X)$ is a p.d.f.