Let $a_n = \dfrac{\cos^2(n)}{2^n}$

$$-1 \leq \cos(n) \leq 1 \quad \text{for all } n$$

$$0 \leq \cos^2(n) \leq 1 \quad \text{for all } n$$

$$\frac{0}{2^n} \leq \frac{\cos^2(n)}{2^n} \leq \frac{1}{2^n} \quad \text{for all } n$$

$$\lim_{n\to\infty} \frac{0}{2^n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{2^n} = 0$$

Hence, by the Squeeze theorem

$$\lim_{n\to\infty} \frac{\cos^2(n)}{2^n} = 0.$$

Examine $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Find A and B s.t. $\displaystyle\frac{A}{n} + \frac{B}{n+3} = \frac{1}{n(n+3)}$

$$A(n+3) + Bn = 1$$

If $n = 0 \Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}$

If $n = -3 \Rightarrow -3B = 1 \Rightarrow B = \frac{-1}{3}$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty} \frac{\frac{1}{3}}{n} - \frac{\frac{1}{3}}{n+3}$

Let S_n be the sequence of partial sums

$$S_n = \frac{\frac{1}{3}}{1} - \cancel{\frac{\frac{1}{3}}{4}} + \cancel{\frac{\frac{1}{3}}{2}} - \cancel{\frac{\frac{1}{3}}{5}} + \frac{\frac{1}{3}}{3} - \cancel{\frac{\frac{1}{3}}{6}} + \cancel{\frac{\frac{1}{3}}{4}} - \cancel{\frac{\frac{1}{3}}{7}} + \cdots$$

$$+ \cancel{\frac{\frac{1}{3}}{n-2}} - \frac{\frac{1}{3}}{n+1} + \cancel{\frac{\frac{1}{3}}{n-1}} - \frac{\frac{1}{3}}{n+2} + \cancel{\frac{\frac{1}{3}}{n}} - \frac{\frac{1}{3}}{n+3}$$

$$= \frac{1}{3} + \frac{1}{6} + \frac{1}{9} - \frac{\frac{1}{3}}{n+1} - \frac{\frac{1}{3}}{n+2} - \frac{\frac{1}{3}}{n+3}$$

$$\lim_{n\to\infty} S_n = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{9+3+2}{18} = \frac{14}{18} = \boxed{\frac{7}{9}}$$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ converges.

Examine $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$

Let $p=1$. Since $f(x) = \dfrac{1}{x(\ln(x))}$ is a positive continuous decreasing function on $[1,\infty)$ we can use the integral test.

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \int_{x=2}^{x=\infty} \frac{1}{u}\,du = \ln(\ln(x))\Big|_2^{\infty} = \infty$$

So, If $p=1$ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges

If $p \neq 1$ we can still use the integral test

$$\int_2^{\infty} \frac{1}{x(\ln(x))^p}\,dx = \int_{x=2}^{x=\infty} \frac{1}{u^p}\,du = \frac{u^{1-p}}{1-p}\Big|_{x=2}^{x=\infty}$$

$$= \lim_{a\to\infty} \frac{(\ln(x))^{1-p}}{1-p}\Big|_2^{a}$$

$$= \begin{cases} \infty & \text{if } p<1 \\ \dfrac{-1}{(1-p)(\ln(2))^{p-1}} & \text{if } p>1 \end{cases}$$

$$\boxed{\text{So } \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p} \ \text{diverges if } p \leq 1, \ \text{converges if } p>1}$$

Examine $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Trick: Use the limit comparison test

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test.

Examine $$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{a\to 0} \frac{\sin(a)}{a}$$

$= 1$ a <u>known</u> limit.

Hence, by the limit comparison test

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges.

Examine $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

Step 1: Show $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges

① Since $\ln(x)$ is strictly increasing

$$\ln(n+1) \geq \ln(n)$$

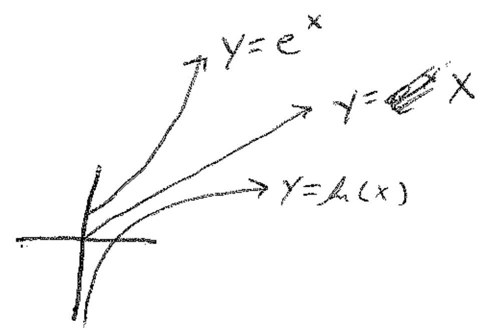
$$\frac{1}{\ln(n+1)} \leq \frac{1}{\ln(n)}$$

So, $b_{n+1} \leq b_n$. Thus the b_n are decreasing

② $\displaystyle\lim_{n \to \infty} \frac{1}{\ln(n)} = 0$

So, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges by the alternating series test.

Step 2: Show $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges.



Since $\ln(n) \leq n$ for all n.

$$\frac{1}{\ln(n)} \geq \frac{1}{n} \text{ for all } n$$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-test, then

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges by direct comparison test.

So, since $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges

then $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges conditionally.

Examine $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$

Look at $\lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2}{(kn+k)!}}{\frac{(n!)^2}{(kn)!}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(kn+k)(kn+k-1)\cdot \cdots \cdot (kn+1)}$

$$= \begin{cases} \frac{1}{4} = \frac{1}{k^2} & \text{if } k=2 \\ 0 & \text{if } k>2 \\ \infty & \text{if } k=0 \text{ or } 1 \end{cases}$$

Hence $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ diverges if $k=0$ or 1
converges if $k \geq 2$.

Examine the power series $\sum_{n=1}^{\infty} \frac{n(2x)^n}{(n^2+1)3^n} = \sum_{n=1}^{\infty}\left(\frac{n}{n^2+1}\right)\left(\frac{2}{3}\right)^n x^n$

Step I: Find Radius of convergence:

$$R = \lim_{n\to\infty} \left| \frac{\frac{n}{n^2+1}\left(\frac{2}{3}\right)^n}{\frac{n+1}{(n+1)^2+1}\left(\frac{2}{3}\right)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n(n+1)^2+n)3}{(n^2+1)(n+1)2} \right|$$

$$= \frac{3}{2}$$

Step 2: Let $x = -\frac{3}{2}$ and $\frac{3}{2}$

If $x = \frac{3}{2}$ we get $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

$\cancel{n \geq 1}$

$\cancel{n^2 \geq n}$

$\cancel{n^2+1 \geq n}$

Look at $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{n^2}$

$= 1$

So $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by limit comparison test.

If $x = -\frac{3}{2}$ we get $\sum_{n=1}^{\infty} \frac{n}{n^2+1}(-1)^n$

This ~~diverges~~ converges by alternating series test.

So, the interval of convergence is $\left[-\frac{3}{2}, \frac{3}{2}\right)$

Let $f(x) = e^x$.

Then $R_5(x)$ is the error when approximating

e^x by $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$.

$f^{(6)}(t) = e^t$, So, $|f^{(6)}(t)| \le e^{-2}$ on $[-4, -2]$

By Taylors Theorem

$$|R_5(x)| \le \frac{e^{-2}|x|^6}{6!} \quad \text{on } [-4,-2]$$

So, $\boxed{|R_5(x)| \le \frac{e^{-2}4^6}{6!}}$ on $[-4, -2]$.