3. (10 pts) Use Taylor series to find the following limit

$$lim_{x \to 0} \frac{e^x - 1 - x}{cos(x) - 1}$$

$$\lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - 1 - x}{\cancel{1 - \frac{x^3}{6} + \frac{x^5}{5!}} \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 1}$$

$$\lim_{x \to 0} \frac{\frac{x^2}{2} + \frac{x^3}{6} + \cdots}{-\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

$$\lim_{x \to 0} \frac{\frac{1}{2} + \frac{x}{6} + \cdots}{-\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots} = \frac{\frac{1}{2}}{-\frac{1}{2}} = \boxed{-1}$$

4. (10 pts) Evaluate

$$\int x^2 e^x dx$$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x$$

$u = x^2 \qquad u' = 2x$

$v' = e^x \qquad v = e^x$

$u = 2x \qquad u' = 2$

$v' = e^x \qquad v = e^x$

$$= x^2 e^x - 2x e^x + \int 2e^x dx$$

$$= \boxed{x^2 e^x - 2x e^x + 2e^x + C}$$

6. (10 pts) A tank contains 5kg of salt dissolved in 500L of water. Brine that contains 2kg of salt per liter is pumped in at a rate of 3L per minute. The solution is thoroughly mixed and drained at a rate of 3L per minute. What is the amount of salt in the tank at t minutes?

$y(t)$ = amount of salt in kg $\qquad y(0) = 5$

$$y' = \left(\begin{matrix}\text{consen}\\ \text{in}\end{matrix}\right)\left(\begin{matrix}\text{flow}\\ \text{in}\end{matrix}\right) - \left(\begin{matrix}\text{consen}\\ \text{out}\end{matrix}\right)\left(\begin{matrix}\text{flow}\\ \text{out}\end{matrix}\right)$$

$$y' = \left(2\,\frac{kg}{L}\right)\left(3\,\frac{L}{min}\right) - \left(\frac{y(t)}{500}\right)(3)$$

$$y' = 6 - \frac{3}{500}\,y(t)$$

$$y' + \frac{3}{500}\,y(t) = 6$$

$$v(t) = e^{\int \frac{3}{500}dt} = e^{\frac{3}{500}t}$$

$$y e^{\frac{3}{500}t} = \int e^{\frac{3}{500}t}\cdot 6\,dt$$

$$y e^{\frac{3}{500}t} = 1000\,e^{\frac{3}{500}t} + C$$

$$y = 1000 + C e^{\frac{-3}{500}t}$$

$$5 = 1000 + C$$

$$C = -995$$

$$\boxed{y = 1000 \cancel{+} - 995\,e^{\frac{-3}{500}t}}$$

4. Evaluate

$$\int \frac{1}{x(x^2-1)}dx$$

$$\frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{x-1}$$

$$1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

$1 = -A$ $\quad$ $1 = B(-1)(-2)$ $\quad$ $1 = C(1)(2)$

$A = -1$ $\quad$ $B = 1/2$ $\quad$ $C = 1/2$

$$\int \frac{1}{x(x^2-1)}dx = \int \frac{-1}{x}dx + \int \frac{1/2}{x+1}dx + \int \frac{1/2}{x-1}dx$$

$$= \boxed{-\ln|x| + \tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x-1| + C}$$

5. (10 pts) Evaluate

$$\int sin^5(x)dx$$

$$\int \sin^2(x) \circ \sin^2(x) \circ \sin(x)dx$$

$$= \int (1-\cos^2(x))(1-\cos^2(x)) \sin(x)dx$$

$$= \int (1 - 2\cos^2(x) + \cos^4(x)) \sin(x)dx$$

$$= -\int 1 - 2u^2 + u^4 du$$

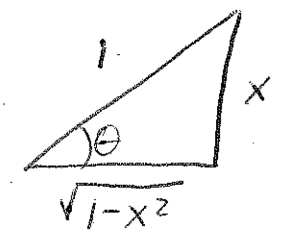
$u = \cos(x)$
$du = -\sin(x)dx$

$$= -u + \frac{2}{3}u^3 - \frac{u^5}{5} + C$$

$$= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C$$

6. (10 pts) Evaluate. (Hint: Use the trig substitution method.)

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx$$



$$x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta \, d\theta$$

$$\frac{1}{\sqrt{1-x^2}} = \sec\theta$$

$$\int \frac{1}{(1-x^2)^{3/2}} dx = \int \sec^3\theta \cos\theta \, d\theta$$

$$= \int \sec^2\theta \, d\theta$$

$$= \tan\theta + C$$

$$= \boxed{\frac{x}{\sqrt{1-x^2}} + C}$$

7. (5 pts) **This is a challenging problem, leave it until last.** Solve the following D.E.

$$ln(ln(y))y' = yln(y)sec^2(x)tan(x)$$

$$\int \frac{\ln(\ln(y))}{y\ln(y)}\,dy = \int \sec^2(x)\tan(x)\,dx$$

$u = \ln(y)$ $\qquad\qquad z = \sec(x)$

$du = \frac{1}{y}\,dy$ $\qquad\qquad dz = \sec(x)\tan(x)dx$

$$\int \frac{\ln(u)}{u}\,du = \int z\,dz$$

$v = \ln(u)$

$dv = \frac{1}{u}\,du$

$$\int v\,dv = \frac{z^2}{2} + c$$

$$\frac{v^2}{2} + c = \frac{1}{2}\sec^2(x) + c$$

$$\frac{1}{2}(\ln(\ln(y)))^2 \cancel{+c} = \frac{1}{2}\sec^2(x) + c$$