Math 104: Series and Approximations II

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Taylor's Formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x)$ is the **error term of order n**.

Theorem (Taylor's Theorem)

Given a Taylor Series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$, *if there is a constant* M *such that* $|f^{(n+1)}(t)| < M$ *for all* t *between* a *and* x, *then* $|R_n(x)| < M\frac{|x-a|^{n+1}}{(n+1)!}$

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Estimate the error for approximating $cos(x)$ on $[-2\pi, 2\pi]$ using the first four terms of its Maclaurin Series.

Estimating a Series

Given a series $\Sigma_{i=0}^{\infty} a_i$ with a sequence of partial sums $\{S_n\}_{n=1}^{\infty}$, we know that the **Error** in estimating the series by the n-th partial sum is

$$R_n = (\Sigma_{i=0}^{\infty} a_i) - S_n = \Sigma_{i=n+1}^{\infty} a_i$$

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We can use integrals to estimate R_n.

Theorem

If $\Sigma_{i=0}^{\infty} a_i$ converges and $f(x)$ is a positive, decreasing, continuous function such that $f(i) = a_i$ for all i, then

$$\int_{n+1}^{\infty} f(x) \leq R_n \leq \int_{n}^{\infty} f(x)$$

Example

1. Approximate $\sum_{i=0}^{\infty} \frac{1}{i^3}$ using the first 10 terms of the series.
2. Estimate R_{10}.
3. Use 1 and 2 to better estimate $\sum_{i=0}^{\infty} \frac{1}{i^3}$.
4. How many terms are required to ensure that the sum is accurate with in .0005?

Estimating Alternating Series

Given an alternating series $\Sigma_{i=0}^{\infty}(-1)^i b_i$, estimating the error is much easier

Theorem

Given an alternating series $\Sigma_{i=0}^{\infty}(-1)^i b_i$ such that

1. $b_{i+1} \leq b_i$ *for all i.*
2. $lim_{i\to\infty} b_i = 0$

Then

$$|R_n| \leq b_{n+1}$$

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Example: How many terms of the series $\Sigma_{i=0}^{\infty}\frac{(-1)^i}{i^3}$ are required to ensure that the sum is accurate with in .0005?