Math 104: Power Series and Approximations

Ryan Blair

University of Pennsylvania

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Power Series

Definition

A **Power Series** is a series and a function of the form

$$P(x) = \Sigma_{k=0}^{\infty} c_k (x-a)^k = c_1 + c_2(x-a) + c_3(x-a)^2 + \ldots$$

where x is a variable, the c_i are constants and we say $P(x)$ is centered at a.

Let R be the radius of convergence of $P(x)$.

$$R = lim_{k\to\infty} \left|\frac{c_k}{c_{k+1}}\right|$$

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If the radius of convergence of a Taylor series is R, find the radius of convergence of an antiderivative and the derivative.

Taylor's Formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x)$ is the **error term of order n**.

Theorem (Taylor's Theorem)

Given a Taylor Series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$, *if there is a constant* M *such that* $|f^{(n+1)}(t)| < M$ *for all* t *between* a *and* x, *then* $|R_n(x)| < M\frac{|x-a|^{n+1}}{(n+1)!}$

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Uses: Can show Taylor series converges if $|R_n(x)|$ goes to zero as n goes to infinity, Can get estimates for functions.

Examples

1. Show that the Maclaurin series for $cos(x)$ converges to $cos(x)$ for all x using Taylor's Theorem.
2. Show that the Maclaurin series for $\frac{1}{1-x}$ converges to $\frac{1}{1-x}$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$ using Taylor's Theorem.
3. Estimate the error for approximating e^x on $[-2, 2]$ using the first four terms of its Maclaurin Series.
4. Estimate the error for approximating $cos(x)$ on $[-2\pi, 2\pi]$ using the first four terms of its Maclaurin Series.