Math 104: Power Series

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Rules for combining series

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$$\sum_{i=1}^{\infty} a_i + b_i = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i.$$

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If $\Sigma_{i=1}^{\infty} a_i$ and $\Sigma_{i=1}^{\infty} b_i$ converge, $a_i > 0$ and $b_i > 0$ for all i, then $\Sigma_{i=1}^{\infty} (a_i)(b_i)$ converges.

Abstracting Taylor Series to Power Series

Definition

The **Taylor series** generated by a function f at $x = a$ is

$$\Sigma_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots$$

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A **Power Series** is a series and a function of the form

$$P(x) = \Sigma_{k=0}^{\infty} c_k (x-a)^k = c_1 + c_2(x-a) + c_3(x-a)^2 + \ldots$$

where x is a variable, the c_i are constants and we say $P(x)$ is centered at a.

For what values of x does a power series converge?

Convergence of Power Series

A power series $\sum_{k=0}^{\infty} c_k (x-a)^k$ fits into one of the following three categories:

1. It converges for all x
2. It converges only at $x = a$
3. It converges for all x such that $|x-a| < R$ where R is some positive constant and may or may not converge at $x = a + R$ and $x = a - R$.

In the third case R is called the **radius of convergence**.

Ratio Test

Theorem

Given a series $\sum_{i=1}^{\infty} a_i$. *If*

$$lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = L,$$

then

1. *If* $L < 1$, *the series converges absolutely.*
2. *If* $L = 1$, *the test is inconclusive.*
3. *If* $L > 1$, *the series diverges.*

How to find the radius of convergence.

Using the ratio test we see that $\Sigma_{k=0}^{\infty} c_k(x-a)^k$ converges if

$$lim_{k\to\infty}|\frac{c_{k+1}(x-a)}{c_k}| < 1$$

So, we get

$$R = lim_{k\to\infty}|\frac{c_k}{c_{k+1}}|$$

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If the radius of convergence of a Taylor series is R, find the radius of convergence of an antiderivative and the derivative.

Interval of Convergence

Given a power series $\Sigma_{k=0}^{\infty} c_k(x-a)^k$ with radius of convergence R, the **interval of convergence** is one of the following where we include endpoints it the series convergence at those points.

$$(a-R, a+R), [a-R, a+R), (a-R, a+R], [a-R, a+R]$$

Find the interval of convergence of $\Sigma_{k=0}^{\infty} \frac{(x)^k}{k}$.