Math 104: Sequences

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Motivating Example

We need to be careful with infinite lists of numbers

$$ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots$$

So, what is the following rearrangement equal to?

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \ldots$$

Sequences

Definition

A **sequence** is an ordered set of real numbers, equivalently, a **sequence** is an function from the positive integers to the real numbers.

We denote the terms of a sequence by $a_1, a_2, a_3, a_4, \ldots$ and the **general** term or the **n-th** term of a sequence is labeled a_n.

Presentation of Sequences

A sequence may be given as a **formula**

$$a_n = \frac{n}{n+1}$$

or as a recursive definition

$$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$$

Limits of Sequences

Thinking of a sequence as a function $f : \mathbb{Z}^+ \to \mathbb{R}$ we can take a limit $lim_{n\to\infty} a_n = L$

Theorem

If $f : \mathbb{R} \to \mathbb{R}$, $f(n) = a_n$ for all $n \in \mathbb{Z}^+$ and $lim_{x\to\infty} f(x) = L$, then $lim_{n\to\infty} a_n = L$

Operations with Limits

If $a_n \to a$ and $b_n \to b$, then
$a_n \pm b_n \to a \pm b$
$ca_n \to ca$
$a_n \times b_n \to a \times b$
$\frac{a_n}{b_n} \to \frac{a}{b}$

Theorem

(Squeeze) Given sequences a_n, b_n and c_n such that $a_n \leq b_n \leq c_n$ for all n and $lim_{n\to\infty} a_n = lim_{n\to\infty} c_n = L$, then

$$lim_{n\to\infty} b_n = L$$

Convergence and Divergence

Examples of sequences that diverge

$$a_n = (-1)^n$$

$$a_n = 2^n$$

Exercise: If $r \in \mathbb{R}$, when does $a_n = r^n$ converge and diverge? (this is called a geometric sequence)

Alternating Sequences

An **alternating** sequence is of the form $a_n = (-1)^n b_n$ where $b_n \geq 0$ for all n.

Theorem

Given an alternating sequence a_n, if $lim_{n\to\infty}|a_n| = 0$ then $lim_{n\to\infty} a_n = 0$.

Exercise: Prove the above theorem using our limit rules and the squeeze theorem.

Monotonic Sequences

Definition

A sequence is **increasing** if $a_n \leq a_{n+1}$ for all n.
A sequence is **decreasing** if $a_n \geq a_{n+1}$ for all n.
If a sequence is decreasing or increasing we say it is **monotonic**.

Definition

A sequence is **bounded above** if there exists a constant M such that $a_n \leq M$ for all n.
A sequence is **bounded below** if there exists a constant m such that $a_n \geq m$ for all n.
A sequence is **bounded** if it is both bounded above and bounded below.

Monotonic Sequences

Theorem

Every bounded monotonic sequence converges