Math 104: Improper Integrals

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Outline

Improper integrals

Definite integrals $\int_a^b f(x)dx$ were required to have

- finite domain of integration $[a, b]$
- finite integrand $f(x) < \pm\infty$

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Improper integrals

1. Infinite limits of integration
2. Integrals with vertical asymptotes i.e. with infinite discontinuity

Infinite limits of integration

Definition

Improper integrals are said to be

- convergent if the limit is finite and that limit is the value of the improper integral.
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If as a limit the improper integral is finite we say the integral **converges**, while if the limit is infinite or does not exist, we say the integral **diverges**.

Convergence is good (means we can do the integral); divergence is bad (means we can't do the integral).

Example 1

Find

$$\int_0^\infty e^{-x}\,dx.$$

(if it even converges)

Example 2

Find

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

(if it even converges)

Example 3, the p-test

The integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

1. Converges if $p > 1$;
2. Diverges if $p \le 1$.

Convergence vs. Divergence

In each case, if the limit exists (or if both limits exist, in Example 2!), we say the improper integral **converges**.

If the limit fails to exist or is infinite, the integral **diverges**. In Example 2, if either limit fails to exist or is infinite, the integral diverges.

Example 4

Find

$$\int_0^2 \frac{2x}{x^2 - 4}\, dx.$$

(if it converges)

Example 5

$$\text{Find} \int_0^3 \frac{1}{(x-1)^{2/3}}\, dx, \quad \text{if it converges.}$$

Solution:

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Solution: We might think just to do

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = \left[3(x-1)^{1/3}\right]_0^3,$$

Example 5

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Solution: We might think just to do

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = \left[3(x-1)^{1/3}\right]_0^3,$$

but this is not okay!

Tests for convergence and divergence

The gist:

1. If you're smaller than something that converges, then you converge.
2. If you're bigger than something that diverges, then you diverge.

Theorem

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\, dx$ *converges if* $\int_a^\infty g(x)\, dx$ *converges.*
2. $\int_a^\infty g(x)\, dx$ *diverges if* $\int_a^\infty f(x)\, dx$ *diverges.*

Example 6

Which of the following integrals converge?

$$(a)\ \int_1^{\infty} e^{-x^2}\ dx, \qquad (b)\ \int_1^{\infty} \frac{\sin^2(x)}{x^2}\ dx.$$

Limit Comparison Test

Theorem

If positive functions f and g are continuous on $[a, \infty)$ *and*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\ dx \quad \text{and} \quad \int_a^\infty g(x)\ dx$$

BOTH converge or BOTH diverge.

Example 7: Let $f(x) = \frac{1}{\sqrt{x}+1}$; consider

$$\int_1^\infty \frac{1}{\sqrt{x}+1}\ dx.$$

Does the integral converge or diverge?