Math 104: Centroids and Centers of Mass

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Outline

Averages

Definition

The average of $x_1, x_2, ..., x_n$ is given by

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

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The average of a function $f(x)$ on and interval $[a, b]$ is given by

$$\overline{f} = \frac{\int_a^b f(x)dx}{b - a}$$

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Example: Find the average of $sin(x)$ on $[0, \pi]$.

Center of mass of particles

Let $(x_1, y_1), (x_2, y_2), ...(x_n, y_n)$ be particles in the plane with masses $m_1, m_2, ..., m_n$ respectively.

Then their center of mass is the point $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}$$

$$\overline{y} = \frac{\sum_{i=1}^{n} y_i m_i}{\sum_{i=1}^{n} m_i}$$

Centroid

The centroid is the point at which an object **constructed of uniform material** would balance. This is different than center of mass.

Definition

(Intuitive) The centroid of a planar region $R = \{(x, y)|a \leq x \leq b, g(x) \leq y \leq f(x)\}$ is the point $(\overline{x}, \overline{y})$ given by the average values of x and y over R.

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Example: Find the centroid of the interval from a to b using the notion of integrals as averages

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$$\overline{x} = \frac{\int_a^b x(f(x) - g(x))dx}{\int_a^b f(x) - g(x)dx}$$

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Example: Centroid of the portion of the unit disk in the first quadrant

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Example: Centroid of the region between $y = 4 - x^2$ and the x-axis.

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Example: Centroid of the region between $y = 4 - x^2$ and the x-axis.
Example: Centroid of the region between $y = sin(x)$ and $y = cos(x)$ for $0 \le x \le \frac{\pi}{4}$.

Center of mass

What if the material making up R has a variable density given by $\rho(x)$?

Definition

The center of mass of a planar region $R = \{(x,y) | a \le x \le b, g(x) \le y \le f(x)\}$ with density $\rho(x)$ is the point $(\overline{x}, \overline{y})$ given by,

$$\overline{x} = \frac{\int_a^b x\rho(x)(f(x) - g(x))dx}{\int_a^b \rho(x)(f(x) - g(x))dx}$$

$$\overline{y} = \frac{\int_a^b \frac{1}{2}\rho(x)((f(x))^2 - (g(x))^2)dx}{\int_a^b \rho(x)(f(x) - g(x))dx}$$