Math 104: Taylor series, Limits and l'Hospital's rule

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Outline

Taylor Series

Definition

The **Taylor series** generated by a function f at $x = a$ is

$$\Sigma_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots$$

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To use this formula we need to know the values of ALL the derivatives of f at a value a.

Tricks to finding Taylor Series

Problem: Find the first 3 terms of the Taylor series for $f(x) = cos(x)sin(x)$ at $x = 0$.

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What is the point? Skip tedious derivative calculations

Manipulating Series

Definition

The **geometric series** for $|x| > 1$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots = \Sigma_{k=0}^{\infty} x^k$$

Ways to formally manipulate Series:

1. Substitution
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3. Integration

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The pay off of all of this work with Taylor series will be a greater understanding of Limits and Derivatives

Revisiting Limits

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If for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - a| < \delta$ then $|f(x) - L| < \epsilon$, we say

$$lim_{x \to a} f(x) = L$$

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Examples Let $f(x) = |x|$. Find $lim_{x \to 0} f(x)$ and $lim_{x \to 0} f'(x)$.

The power of Taylor series when finding limits

Use your knowledge of Taylor series to find the following limits:

$$lim_{x\to 0}\frac{sin(x)}{x}$$

$$lim_{x\to 0}\frac{sin(3x)}{e^x-1}$$

$$lim_{x\to 0}\frac{e^x-1-x}{x^2}$$

l'Hospital's rule

Theorem

If

$$lim_{x \to a} f(x) = 0 = lim_{x \to a} g(x),$$

Then

$$lim_{x \to a} \frac{f(x)}{g(x)} = lim_{x \to a} \frac{f'(x)}{g'(x)}.$$