Math 103: Limits and One-Sided Limits of Functions

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Outline

Motivating Question:

How do we determine the behavior of a function near a point without worrying about its behavior at the point?

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How do we determine the behavior of a function near a point without worrying about its behavior at the point?

Example: How does the following function behave near $x = 1$

$$f(x) = \frac{x^3 - x^2}{x - 1}$$

The Answer is the Limit!

Definition

If $f(x)$ is arbitrarily close to L for all x sufficiently close to x_0, we say f approaches the **limit** L as x approaches x_0 and write:

$$lim_{x \to x_0} f(x) = L$$

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Evaluate

$$lim_{x \to 1} \frac{x^3 - x^2}{x - 1}$$

Limit Laws I

If L,M,a and k are real numbers and

$$lim_{x \to a} f(x) = L \ and \ lim_{x \to a} g(x) = M$$

1. $lim_{x \to a}[f(x) + g(x)] = L + M$
2. $lim_{x \to a}[f(x) - g(x)] = L - M$
3. $lim_{x \to a}[kf(x)] = kL$
4. $lim_{x \to a}[f(x)g(x)] = LM$
5. $lim_{x \to a}\frac{f(x)}{g(x)} = \frac{L}{M} \ \ M \neq 0$
6. $lim_{x \to a}[f(x)]^n = L^n, \ \ n \epsilon \mathbb{Z}^+$
7. $lim_{x \to a}[f(x)]^{\frac{1}{n}} = L^{\frac{1}{n}}, \ \ n \epsilon \mathbb{Z}^+$

Some Limit Theorems

Theorem

Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, *then*

$$lim_{x \to c} P(x) = a_n c^n + a_{n-1} c^{n-1} + ... + a_0 = P(c)$$

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Theorem

If $Q(x)$ *and* $P(x)$ *are polynomials and* $Q(c) \neq 0$, *then*

$$lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

Evaluating Limits Involving Zeros in the Denominator

Sometimes we can still use the equality

$$lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

even if $Q(c) = 0$

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Evaluate:

$$lim_{x \to 0} \frac{x^3 + x}{x^2 - x}$$

The Sandwich Theorem

Theorem

If $f(x) \leq g(x) \leq h(x)$ *when* x *is near* c *and*

$$lim_{x \to c} f(x) = lim_{x \to c} h(x) = L$$

$$then \ lim_{x \to c} g(x) = L$$

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Evaluate:

$$lim_{x \to 0}\ x\ sin(\frac{1}{x})$$

Definition of One-Sided Limit

Definition

If $f(x)$ is arbitrarily close to L for all x sufficiently close to c and greater than c, we say f approaches the **rigth-hand limit** L as x approaches c and write:
$lim_{x \to c^+} f(x) = L$

Definition

If $f(x)$ is arbitrarily close to L for all x sufficiently close to c and less than c, we say f approaches the **left-hand limit** L as x approaches c and write:
$lim_{x \to c^-} f(x) = L$

Theorem

$$lim_{x \to c} f(x) = L$$

if and only if

$$lim_{x \to c^+} f(x) = L \text{ and } lim_{x \to c^-} f(x) = L.$$