Math 103: The Fundamental Theorem of Calculus

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Outline

Definition

The **area** A of a region S that lies under the graph of a continuous function f is the limit of the sum of areas of the approximating rectangles:

$$A = lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

Theorem

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and the c_i are a collection of sample points.

Properties of Integrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b c \;\; dx = c(b-a)$ where c is any constant.
4. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ where c is a constant.
6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ if $a \leq c \leq b$.

Theorem

(Fundamental Theorem of Calculus, Part 1) If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \;\; a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

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Theorem

(Fundamental Theorem of Calculus, Part 2) If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where F is any antiderivative of f, that is, a function such that $F' = f$.

Net Change

Theorem

The next change of a function $F(x)$ over an interval $[a, b]$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)dx$$

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Exercise Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.

Area under the curve

To find the area between the graph $y = f(x)$ and the x-axis on the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of $f(x)$.
2. Integrate f over each interval.
3. Add the absolute value of the intervals.