1. Practice Midterm 1

Problem 1. At what value(s) of x is the following function discontinuous?

$$f(x) = \begin{cases} x^2 + 4x + 5 & : if \ \ x < -2 \\ \frac{1}{2}x & : if \ \ -2 \le x \le 2 \\ 1 + \sqrt{x-2} & : if \ \ x > 2 \end{cases}$$

Since polynomial and root functions are continuous everywhere on their domain, we only need to check continuity at $x = -2$ and $x = 2$.

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} x^2 + 4x + 5 = (-2)^2 + 4(-2) + 5 = 1$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \tfrac{1}{2}x = \tfrac{1}{2}(-2) = -1$$

Since $\lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$, then $f(x)$ is discontinuous at $x = -2$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \tfrac{1}{2}x = \tfrac{1}{2}(2) = 1$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 1 + \sqrt{x-2} = 1 + \sqrt{2-2} = 1$$

$$f(2) = \tfrac{1}{2}(2) = 1$$

Hence $f(x)$ is continuous at $x = 2$.

Problem 2. Find the derivative of $f(x) = \sqrt{x^2+1}$ using the **limit definition** of derivative.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{(x+h)^2+1} - \sqrt{x^2+1}\right)}{h} \frac{\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}{\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h \to 0} \frac{(x+h)^2+1 - (x^2+1)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h \to 0} \frac{x^2+2xh+h^2+1-x^2-1}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h \to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}$$

$$= \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

Problem 3. Prove the following product rule for derivatives

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) \circ \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} + \lim_{x \to h} g(x) \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= f(x) \circ \frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

Problem 4. What is the slope of the tangent line to $f(x) = (x^2)(e^x)$ at $x = 2$?

$$f'(x) = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)$$

$$= x^2 e^x + 2x e^x$$

$$f'(2) = 2^2 e^2 + 2(2) e^2$$

$$= 8e^2$$

The slope of the line tangent to $f(x)$ at $x=2$ is $\boxed{8e2}$

Problem 5. Find the value of the limit.

$$lim_{x \to 2} \frac{\sqrt{x+7}-3}{(x-2)(x+1)}$$

$$\lim_{x \to 2} \frac{\sqrt{x+7}-3}{(x-2)(x+1)} \frac{(\sqrt{x+7}+3)}{(\sqrt{x+7}+3)}$$

$$= \lim_{x \to 2} \frac{x+7-9}{(x-2)(x+1)(\sqrt{x+7}+3)}$$

$$= \lim_{x \to 2} \frac{\cancel{x-2}}{(\cancel{x-2})(x+1)(\sqrt{x+7}+3)}$$

$$= \lim_{x \to 2} \frac{1}{(x+1)\cdot(\sqrt{x+7}+3)}$$

$$= \frac{1}{(2+1)(\sqrt{2+7}+3)}$$

$$= \frac{1}{3(6)}$$

$$= \frac{1}{18}$$

Problem 6. Find the value of the limit.

$$lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{1-3x}$$

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{1-3x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}\sqrt{2x^2+1}}{\frac{1}{x}-3}$$

$$=\lim_{x\to\infty}\frac{\sqrt{2+1/x^2}}{\frac{1}{x}-3}$$

$$=\frac{\sqrt{2+\lim_{x\to\infty}\frac{1}{x^2}}}{\lim_{x\to\infty}\frac{1}{x}-3}$$

$$=\frac{\sqrt{2}}{-3}$$

$$=\boxed{-\frac{\sqrt{2}}{3}}$$

Problem 7. Find values a and b such that $f(x)$ is differentiable everywhere

$$f(x) = \begin{cases} x^{\frac{1}{2}} + x & : if \;\; x \geq 1 \\ ax^2 + bx + 1 & : if \;\; x < 1 \end{cases}$$

Since polynomials and root functions are differentiable everywhere on their domain, we only need to check where $x = 1$.

1) Need $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$$\lim_{x \to 1^-} ax^2 + bx + 1 = \lim_{x \to 1^+} x^{1/2} + x$$

$$a(1)^2 + b(1) + 1 = 1^{1/2} + 1$$

$$\boxed{a + b = 1} \; (*)$$

2) Need $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$

$$\frac{d}{dx}(ax^2 + bx + 1)(1) = \frac{d}{dx}(x^{1/2} + x)(1)$$

$$2a(1) + b(1) = \frac{1}{2}(1)^{-1/2} + 1$$

$$\boxed{2a + b = \frac{3}{2}}$$

3) Find a and b

by $(*)$ $a = 1 - b$, so $2(1-b) + b = \frac{3}{2}$

$$2 - b = \frac{3}{2}$$

$$\boxed{\begin{array}{c} b = 1/2 \\ a = 1/2 \end{array}}$$

Problem 8. Find all points on the graph of $f(x) = x^3 - 2x$ where the tangent line has slope 1.

$f'(x) = 3x^2 - 2$

Find all x s.t. $f'(x) = 1$

$1 = 3x^2 - 2$

$3 = 3x^2$

$x = \pm 1$

$f(1) = 1^3 - 2(1) = -1$

$f(-1) = (-1)^3 - 2(-1) = 1$

$\boxed{(1, -1) \text{ and } (-1, 1)}$

Problem 9. Evaluate the following limit.

$$lim_{x\to 0}\frac{xcsc(2x)}{cos(5x)}$$

$$\lim_{x\to 0}\frac{x\csc(2x)}{\cos(5x)} = \lim_{x\to 0}\frac{x}{\sin(2x)}\cdot\frac{1}{\cos(5x)}$$

$$= \lim_{x\to 0}\frac{x}{\sin(2x)}\cdot\lim_{x\to 0}\frac{1}{\cos(5x)}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{2x}{\sin(2x)}\cdot\lim_{x\to 0}\frac{1}{\cos(5x)}$$

$$= \frac{1}{2}\cdot(1)\cdot\frac{1}{\cos(5\cdot 0)}$$

$$= \boxed{\frac{1}{2}}$$