MATH 103 - Fall 2011
Practice Midterm Two

Name:

TA:

Recitation Number:

You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. Show all work, even on multiple choice or short answer questions—I will be grading as much on the basis of work shown as on the end result. Include all units. Remember to put your name at the top of this page. Good luck.

Problem	**Score (out of)**
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
6	(10)
7	(10)
8	(10)
9	(10)
Total	(90)

1. (10 points) Find the derivative of the following function.

$$f(x) = ln|tan^{-1}(\frac{e^x}{x})|$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

Hence, by the chain rule

$$\frac{d}{dx}(f(x)) = \frac{1}{\tan^{-1}(\frac{e^x}{x})} \cdot \frac{d}{dx}\left(\tan^{-1}\left(\frac{e^x}{\cancel{x}}\right)\right)$$

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

Hence, again by chain rule

$$\frac{d}{dx}(f(x)) = \frac{1}{\tan^{-1}(\frac{e^x}{x})} \cdot \frac{1}{1+(\frac{e^x}{x})^2} \cdot \frac{d}{dx}\left(\frac{e^x}{x}\right)$$

By quotient rule, we conclude.

$$\boxed{\frac{d}{dx}(f(x)) = \frac{1}{\tan^{-1}(\frac{e^x}{x})} \cdot \frac{1}{1+(\frac{e^x}{x})^2} \cdot \frac{x e^x - e^x}{x^2}}$$

2.(10 points) Use the Mean Value Theorem to prove the following theorem

If $f(x)$ and $g(x)$ are everywhere differentiable functions such that $f'(x) = g'(x)$, then there exists a constant C such that $f(x) = g(x) + C$.

Let x_1, x_2 be real numbers s.t. $x_1 < x_2$.

Since $f(x)$ and $g(x)$ are differentiable everywhere, then $h(x) = f(x) - g(x)$ is differentiable everywhere. Hence $h(x)$ satisfies the hypotheses of the Mean Value Thm on $[x_1, x_2]$. Therefore,

$$\frac{h(x_2) - h(x_1)}{x_2 - x_1} = h'(c) \text{ for some } c \text{ s.t. } x_1 < c < x_2.$$

But, we know $h'(x) = f'(x) - g'(x) = 0$.

So, $h'(c) = 0$.

By substitution $\dfrac{h(x_2) - h(x_1)}{x_2 - x_1} = 0$

Since $x_2 - x_1 \neq 0$ $\quad h(x_2) - h(x_1) = 0$

$$h(x_2) = h(x_1)$$

Since x_2 and x_1 are arbitrary

$h(x) = h(x_1) = C$ for all real numbers x.

Thus, $f(x) - g(x) = C$. So $f(x) = g(x) + C$. $\square$

3.(10 points) A continuous and differentiable function f decreases steadily on the interval $[-4, 1]$, and satisfies $f(-4) = 1$ and $f(1) = -1$. Which of the following **CANNOT** be true? Circle **ALL** that apply, and justify your answers.

(A) f has a local maximum at some c in $(-4, 1)$.

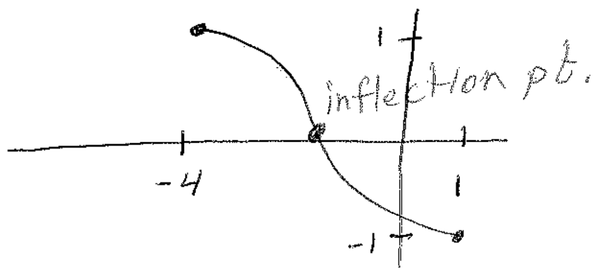
(B) f has an infection point at some c in $(-4, 1)$.

(C) $f'(x) \leq -2$ for all x in $[-4, 1]$.

(D) $f'(x) \geq -1$ for all x in $[-4, 1]$.

(A) Is never true since a differentiable function has a max iff it switches from increasing to decreasing at some point.

(B) Is possible



(C) Is not possible since the MVT would give

$$\frac{f(1) - f(-4)}{1 - (-4)} = f'(c) \leq -2$$

$$\frac{(-1) - (1)}{1 + 4} \leq -2$$

$$\frac{-2}{5} \leq -2 \quad \text{Impossible!}$$

(D) Let $f(x)$ be the line from $(-4, 1)$ to $(1, -1)$. Then $f'(x) = \frac{-2}{5}$, which is greater than or equal to -1.

4.(10 points) Let $f(x) = 3x^4 + 4x^3 - 12x^2 + 3$. How many zeros does the function f have in the interval $[1,3]$? For credit you must justify your answer. **Hint:** Where is $f(x)$ increasing?

$$f(1) = 3(1)^4 + 4(1)^3 - 12(1) + 3$$
$$= 3 + 4 - 12 + 3 = -2$$
$$f(3) = 3(3)^4 + 4(3)^3 - 12(3) + 3$$
$$= 243 + 108 - 36 + 3$$
$$= 318$$

Since f is continuous on $[1,3]$ and $f(1) < 0$ and $f(3) > 0$ then, by the Intermediate value theorem there is a value c in $[1,3]$ s.t. $f(c) = 0$.

Suppose, to form a contradiction, that $f(x)$ has two zeros in the interval $[1,3]$ at values a_1 and a_2.

Since f is differentiable on $[1,3]$ and $f(a_1) = f(a_2) = 0$ then, by Rolle's theorem, there is a value d in $(1,3)$ s.t. $f'(d) = 0$.

However, $f'(x) = 12x^3 + 12x^2 - 24x$
$$= 12x(x^2 + x - 2)$$
$$= 12x(x+2)(x-1)$$

Thus $f'(x)$ has <u>no</u> zeros in $(1,3)$,

So, f <u>does not</u> have two roots in $[1,3]$.

5. (10 points) Find the critical points and intervals of concavity of $f(x) = x + cos(x)$ on the interval $[-2\pi, 2\pi]$. What does the second derivative test tell you about the critical points?

$f'(x) = 1 - \sin(x)$

Find the crit. pts. in $[-2\pi, 2\pi]$

① $0 = 1 - \sin(x)$

$\sin(x) = 1$

$x = \frac{\pi}{2}, \frac{-3\pi}{2}$

② where is $f'(x)$ undefined? Nowhere.

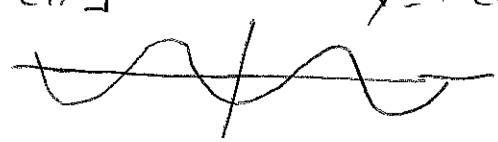
The critical points are $\left(\frac{\pi}{2}, f\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{-3\pi}{2}, f\left(\frac{-3\pi}{2}\right)\right) = \left(\frac{-3\pi}{2}, \frac{-3\pi}{2}\right)$

$f''(x) = -\cos(x)$

$0 = -\cos(x)$ on $[-2\pi, 2\pi]$

$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

$y = -\cos(x)$



$-$ $+$ $-$ $+$ $-$ $f''(x)$

$-\frac{3\pi}{2}$ $-\frac{\pi}{2}$ $\frac{\pi}{2}$ $\frac{3\pi}{2}$

Interval of c-up : $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Interval of c-down : $\left[-2\pi, \frac{-3\pi}{2}\right) \cup \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$

The 2nd-derivative test tells us <u>nothing</u> since $f''(x)$ is zero at the crit pts.

6. (10 points) Find the following limit

$$lim_{x\to\infty}(1+\frac{1}{x})^x$$

Instead look at $\lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right)$

$$= \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

Since $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right) = 0$

we can use L'Hospital's

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1$$

Hence, $\lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right) = 1$.

Examine $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{1}{x}\right)^x\right)}$

$$= e^{\lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right)}$$

$$= e^1 = \boxed{e}$$

7. (10 points) **Carefully** draw a graph for a function $f(x)$ given the following data. Be sure to label all maxima and minima and all asymptotes.

A) The domain of $f(x)$ is $(-\infty, 0) \cup (1, \infty)$.

B) $f(x)$ has no x-intercepts.

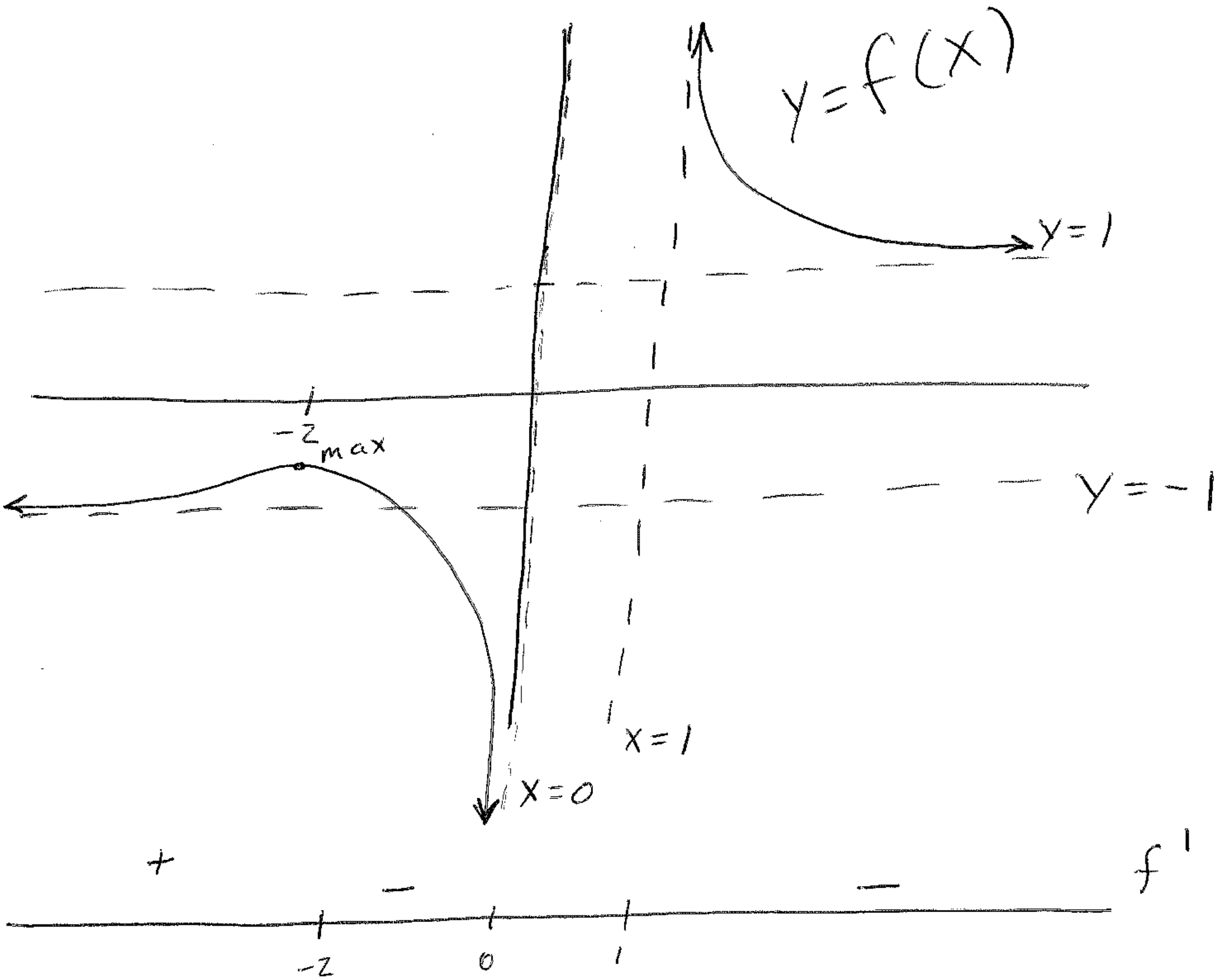
C) The interval of increase of $f(x)$ is $(-\infty, -2)$.

D) The interval of decrease of $f(x)$ is $(-2, 0) \cup (1, \infty)$.

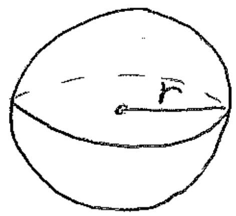
E) $f(x)$ has a critical value at $x = -2$.

F) $f(x)$ has vertical asymptotes at $x = 0$ and $x = 1$.

G) $f(x)$ has horizontal asymptotes $y = 1$ and $y = -1$.



8. (10 points) Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.



V = volume

A = surface area

r = radius

$V = \frac{4}{3}\pi r^3 \qquad A = 4\pi r^2$

Want to find $\frac{dr}{dt}$ if $\frac{dV}{dt}$ is proportional to A.

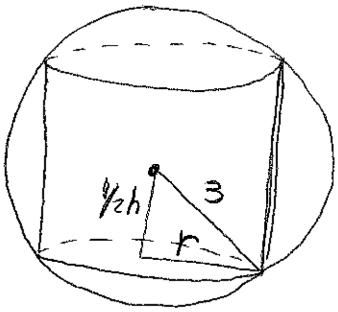
Hence $\frac{dV}{dt} = kA$ for some constant k (this is the definition of proportional)

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

By substitution, $4\pi r^2 \frac{dr}{dt} = k\, 4\pi r^2$.

After cancelling $\boxed{\frac{dr}{dt} = k}$.

9. (10 points) A right circular cylinder is inscribed in a sphere of radius 3 cm. Find the largest possible volume of such a cylinder.

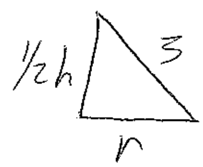


Let V be the volume of the cylinder

h be the height of the cylinder

r be the radius of the base of the cylinder

We want to maximize $V = \pi r^2 h$.

The triangle 

is a right triangle.

So, $9 = \frac{1}{4}h^2 + r^2$, or $r^2 = 9 - \frac{1}{4}h^2$

By substitution $V = \pi(9 - \frac{1}{4}h^2)h$

$$V = 9\pi h - \frac{\pi}{4}h^3$$

$$\frac{dV}{dh} = 9\pi - \frac{3}{4}\pi h^2$$

Find crit. pt.s

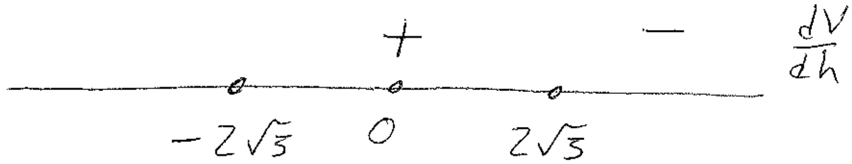
$$0 = 9\pi - \frac{3}{4}\pi h^2$$

$$\frac{3}{4}\pi h^2 = 9\pi$$

$$h^2 = 12$$

$$h = \pm 2\sqrt{3}$$

$\frac{dV}{dh}$ is defined everywhere!



Plug in test pts between crit pts.

$\frac{dV}{dh}(0) = 9\pi$ $\quad \frac{dV}{dh}(4) = -3\pi$

The largest volume is $V(2\sqrt{3}) = \pi(9 - \frac{1}{4}12)(2\sqrt{3}) = \boxed{12\sqrt{3}\pi}\ cm^3$