Math 600 Day 12: More Multilinear Algebra

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Thursday October 21, 2010

Definition

If V is an n-dimensional vector space, define $\Lambda^*(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus ... \oplus \Lambda^n(V)$, where $\Lambda^0(V)$ is defined to be the real numbers. Then $\Lambda^*(V)$ is a vector space.

dim $\Lambda^*(V) = 2^n$ (by summing binomial coefficients).

By extending the wedge product to $\lambda^*(V)$ by linearity, $\Lambda^*(V)$ is an algebra.

It is called the **exterior algebra of forms on V.** If $f : V \to W$ is a linear mapping, note that $f^* : \Lambda^*(W) \to \Lambda^*(V)$ is an algebra homomorphism.

A k-form ω on V is said to be **decomposable** if there are 1-forms $\varphi_1, ..., \varphi_k$ on V such that $\omega = \varphi_1 \wedge ... \wedge \varphi_k$.

Exercise. (a) Let $v_1, ..., v_4$ be a basis for $\mathbb{R}^4$ and let $\varphi_1, ..., \varphi_4$ be the dual basis for $\lambda^1\mathbb{R}^4 = (\mathbb{R}^4)^*$. Show that the 2-form $\omega = \varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4$ is **not** decomposable.

(b) Show that every 2-form on $\mathbb{R}^3$ is decomposable.

A 1-form α on V is just a linear map $\alpha : V \to \mathbb{R}$, so we know perfectly well what is meant by the **kernel** of α:

$$ker(\alpha) = v \epsilon V : \alpha(v) = 0.$$

For example, if $v_1, ..., v_n$ is a basis for V and $\varphi_1, ..., \varphi_n$ is the dual basis, then

$$ker(\varphi_1) = span(v_2, ..., v_n).$$

The kernel of a nonzero 1-form is an $n - 1$ dimensional subspace of V.

Now suppose ω is a 2-form. We define its **kernel** to be

$$ker(\omega) = \{v \epsilon V : \omega(v, w) = 0 \ \ for \ \ all \ \ w \epsilon V\}.$$

For example, in $\mathbb{R}^4$,

$$ker(\varphi_1 \wedge \varphi_2) = span(v_3, v_4)$$

$$ker(\varphi_1 \wedge \varphi_2 + \varphi_3 \wedge \varphi_4) = \{0\}.$$

Likewise, if ω is a k-form, we define

$$ker(\omega) = \{v \epsilon V : \omega(v, w_1, ..., w_{k-1}) = 0 \ \ for \ \ all \ \ w_i \epsilon V\}.$$

Definition

Let v be a vector in V and ω a k-form on V. Then the **interior product** $v\rfloor\omega$ is the $k-1$ form defined by

$$(v\rfloor\omega)(v_1, ..., v_{k-1}) = \omega(v, v_1, ..., v_{k-1}).$$

Differential Forms

We will begin with differential forms on open subsets of Euclidean space $\mathbb{R}^n$, and then extend this to differential forms on smooth manifolds M^n.

Elements of $\mathbb{R}^n$ may be regarded as points p or vectors v.

Fix a point $p \epsilon \mathbb{R}^n$. Then the set of all pairs (p, v), where $v \epsilon \mathbb{R}^n$, will be denoted by $\mathbb{R}^n_p$, and made into a vector space by defining

$$(p, v) + (p, w) = (p, v + w) \;\; and \;\; a(p, v) = (p, av).$$

We call R^n_p the **tangent space** to R_n at the point p, call its elements **tangent vectors**, and visualize them as arrows with their tails at p.

We will usually write (p, v) as v_p.

Pick and fix a basis $e_1, ..., e_n$ for $\mathbb{R}^n$.

Then we get a corresponding basis $(e_1)_p, ..., (e_n)_p$ for each tangent space $\mathbb{R}^n_p$.

A **vector field** V on $\mathbb{R}^n$ is a selection of a tangent vector $V(p)\epsilon\mathbb{R}^n_p$ for each point $p\epsilon\mathbb{R}^n$. We can write

$$V(p) = v^1(p)(e_1)_p + ... + v^n(p)(e_n)_p.$$

The real-valued functions $v^i : \mathbb{R}^n \to \mathbb{R}$ are called the **component functions** of V.

Example. $V = -y\mathbf{i} + x\mathbf{j}$ on $\mathbb{R}^2$.

The vector field is said to be continuous, differentiable, etc. if its component functions v^i are. We will usually deal with C^∞ vector fields, so that we can differentiate the component functions as much as we want, and use the word **smooth** as a synonym for C^∞.

Remark

Vector fields can also be defined on open subsets of $\mathbb{R}^n$ in the same way.

Remark

Vector fields can be added by adding their values at each point, and multiplied by functions likewise.

Differential forms.

In the same spirit as for vector fields, a differential k-form ω on $\mathbb{R}^n$ is a selection of a k-form $\omega(p)\epsilon\Lambda^k(\mathbb{R}^n_p)$ for each point $p\epsilon\mathbb{R}^n$.

If $\varphi_1(p), ..., \varphi_n(p)$ is the dual basis to $(e_1)_p, ..., (e_n)_p$, we can write

$$\omega(p) = \Sigma_{i_1<...<i_k}\omega_{i_1,...,i_k}(p)\varphi_{i1}(p) \wedge ... \wedge \varphi_{ik}(p)$$

for certain coefficient functions $\omega_{i_1,...,i_k} : \mathbb{R}^n \to \mathbb{R}$. We will usually assume that these coefficient functions are of class C^∞.

Differential forms can be defined on open subsets of $\mathbb{R}^n$ in the same way.

Operations on differential forms:

1. addition $\omega + \eta$,
2. wedge product $\omega \wedge \eta$, and
3. multiplication by functions $f\omega$,

are carried out "pointwise".

When it is clear from context that we are talking about differential forms, we will simply call them forms.

Notation.

Now let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function.

Then the derivative of f at each point $p \epsilon \mathbb{R}^n$ is a linear map $f'(p) : \mathbb{R}^n \to \mathbb{R}$. We will think of this as a linear map from the tangent space $\mathbb{R}^n_p$ to $\mathbb{R}$, and write it as $df(p)$. Thus

$$df(p) : \mathbb{R}^n_p \to \mathbb{R} \ \ with \ \ df(p)(v_p) = f'(p)(v).$$

Let $x^i : \mathbb{R}^n \to \mathbb{R}$ be the ith coordinate function. It is a linear map, hence equal to its own derivative, so

$$dx^i(p)(v_p) = v^i.$$

In particular, $dx^i(p)(e_j)_p = \delta^i_j$, so $dx^1(p), ..., dx^n(p)$ is the dual basis to $(e_1)_p, ..., (e_n)_p$.

Thus every differential k-form ω can be written as

$$\omega = \Sigma_{i_1 < ... < i_k} \omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}.$$

Exercise. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, check that the differential 1-form df can be written as

$$df = (\frac{\partial f}{\partial x^1})dx^1 + ... + (\frac{\partial f}{\partial x^n})dx^n.$$