Math 600 Day 11: Multilinear Algebra

Ryan Blair

University of Pennsylvania

Tuesday October 19, 2010

Outline

Multilinear Algebra

V = vector space (typically finite dim'l) over $\mathbb{R}$

V^k = k-fold product $V \times ... \times V$

A function $T : V^k \to \mathbb{R}$ is said to be **multilinear** if it is linear in each variable when the other $k - 1$ variables are held fixed. Such a multilinear function T is called a k-**tensor** on V.

Example. An inner product on V is a 2-tensor which is *symmetric* and *positive definite.*

$\mathcal{T}^k(V) =$ set of all k-tensors on V, is a vector space over $\mathbb{R}$ in the natural way.

Note that $\mathcal{T}^1(V)$ is just the dual space V^*.

If $S \epsilon \mathcal{T}^k(V)$ and $T \epsilon \mathcal{T}^r(V)$, we define a **tensor product** $S \otimes T \epsilon T^{k+r}(V)$ by

$$(S \otimes T)(v_1, ..., v_k, v_{k+1}, ..., v_{k+r}) = S(v_1, ..., v_k) T(v_{k+1}, ..., v_{k+r}).$$

Note that $S \otimes T \neq T \otimes S$.

Tensor Equalities.

$$(S_1 + S_2) \otimes T = (S_1 \otimes T) + (S_2 \otimes T)$$

$$S \otimes (T_1 + T_2) = (S \otimes T_1) + (S \otimes T_2)$$

$$(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$$

$$(S \otimes T) \otimes U = S \otimes (T \otimes U).$$

Exercise. Let $v_1, ..., v_n$ be a basis for V, and let $\varphi_1, ..., \varphi_n$ be the dual basis for $V^* = T^1(V)$, meaning that $\varphi_i(v_j) = \delta_{ij}$. Show that the set of all k-fold tensor products

$$\varphi_{i_1} \otimes ... \otimes \varphi_{i_k}, 1 \leq i_1, ..., i_k \leq n$$

is a basis for $\mathcal{T}^k(V)$, which therefore has dimension n^k.

Definition

If $f : V \to W$ is a linear map, then a linear map $f^* : T^k(W) \to T^k(V)$ is defined by

$$(f * T)(v_1, ..., v_k) = T(f(v_1), ..., f(v_k)).$$

When $k = 1$, this is just the familiar transpose or adjoint of a linear map.

Note that $f^*(S \otimes T) = f^*S \otimes f^*T$.

Definition

A k-tensor $\omega \epsilon \mathcal{T}^k(V)$ is **alternating** if

$$\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k)$$

for all $v_1, ..., v_k \epsilon V$. That is, ω changes sign when exactly two of its variables are interchanged.

An alternating k-tensor is called a **k-form**.

The set of k-forms is denoted by $\Lambda^k(V)$, and is a subspace of $T^k(V)$.

Any k-tensor can be turned into a k-form:

$$Alt(T)(v_1, ..., v_k) =_{defn} (\frac{1}{k!})\Sigma_{\sigma \epsilon S_k}(-1)^{\sigma} T(v_{\sigma(1)}, ..., v_{\sigma(k)}),$$

where S_k is the symmetric group of all permutations of the numbers 1 to k and $(-1)^{\sigma}$ is the sign of the permutation σ.

Exercise. Show that $Alt(T)$ really is alternating.

Other Facts

(a) If T is already alternating, $Alt(T) = T$.

(b) $Alt(Alt(T)) = Alt(T)$.

Definition

If $\omega \epsilon \Lambda^k(V)$ and $\eta \epsilon \Lambda^r(V)$, the **wedge product** $\omega \wedge \eta \epsilon \Lambda^{k+r}(V)$ is defined by

$$\omega \wedge \eta = \frac{(k+r)!}{k!r!} Alt(\omega \otimes \eta).$$

For example, if φ_1 and φ_2 are 1-forms, we have

$$(\varphi_1 \wedge \varphi_2)(v_1, v_2) = \varphi_1(v_1)\varphi_2(v_2) - \varphi_1(v_2)\varphi_2(v_1).$$

Note that for a 1-form φ we have $\varphi \wedge \varphi = 0$.

Exercise. Let ω be a k-form and η an r-form. Show that

$$(\omega \wedge \eta)(v_1, ..., v_k, v_{k+1}, ..., v_{k+r})$$

$$= \Sigma_{\sigma \epsilon S'} (-1)^{\sigma} \omega(v_{\sigma(1)}, ..., v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, ..., v_{\sigma(k+r)}),$$

where S' is the subset of the symmetric group S_{k+r} consisting of all permutations σ such that

$$\sigma(1) < ... < \sigma(k) \;\; and \;\; \sigma(k+1) < ... < \sigma(k+r).$$

Properties of the wedge product:

1. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$
2. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$
3. $(a\omega) \wedge \eta = \omega \wedge (a\eta) = a(\omega \wedge \eta)$
4. $\omega \wedge \eta = (-1)^{kr} \eta \wedge \omega$
5. $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$.

Problem. Show that

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+r+s)!}{k!r!s!} Alt(\omega \otimes \eta \otimes \theta),$$

where $\omega \epsilon \Lambda^k(V)$, $\eta \epsilon \Lambda^r(V)$ and $\theta \epsilon \Lambda^s(V)$.

This is harder.

So now we can drop the parentheses, and simply write $\omega \wedge \eta \wedge \theta$, and likewise for higher order products.

Exercise. Let $v_1, ..., v_n$ be a basis for V , and let $\varphi_1, ..., \varphi_n$ be the dual basis for $V^* = \mathcal{T}^1(V)$. Show that the set of all

$$\varphi_{i1} \wedge ... \wedge \varphi_{ik}, \;\; \textit{with} \;\; 1 \leq i_1 < i_2 < ... < i_k \leq n$$

is a basis for $\Lambda^k(V)$, which therefore has dimension

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Show, in fact, that $(\varphi_{i_1} \wedge ... \wedge \varphi_{i_k})(v_{i_1}, ..., v_{i_k}) = 1$.

Note in particular that $\Lambda^n(V)$ is one-dimensional.