Math 600 Day 10: Lee Brackets of Vector Fields

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Thursday October 14, 2010

Outline

Lie brackets of vector fields.

Let M be a smooth manifold. Then a smooth vector field V on M acts as a "first order differential operator" on smooth functions $f : M \rightarrow \mathbb{R}$ by taking $f \rightarrow V(f) \equiv L_V f$.

If V and W are both smooth vector fields on M, we can use them to operate in succession on smooth functions, taking $f \rightarrow V(W(f))$.

No single vector field can accomplish this composite operation, as is borne out by the appearance of second derivatives in the local coordinate expression of $V(W(f))$.

But if we take $(VW - WV)(f)$, then we claim that there is a single vector field on M which can accomplish the same thing. We call it the **Lie bracket** of V and W, and write

$$[V, W] = VW - WV.$$

To confirm this, we will work in local coordinates and watch the second derivative terms disappear, as follows.

Supposing that $V = v^i \frac{\partial}{\partial x^i}$ and $W = w^i \frac{\partial}{\partial x^i}$, we have

$$
\begin{aligned}
[V, W]f &= V(Wf) - W(Vf) \\
&= v^i \frac{\partial}{\partial x^i}(w^j \frac{\partial f}{\partial x^j}) - w^i \frac{\partial}{\partial x^i}(v^j \frac{\partial f}{\partial x^j}) \\
&= v^i \frac{\partial w^j}{\partial x^i} \frac{\partial f}{\partial x^j} + v^i w^j \frac{\partial^2 f}{\partial x^i \partial x^j} - w^i \frac{\partial v^j}{\partial x^i} \frac{\partial f}{\partial x^j} - w^i v^j \frac{\partial^2 f}{\partial x^i \partial x^j} \\
&= [(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}) \frac{\partial}{\partial x^j}]f.
\end{aligned}
$$

This shows that $[V, W] = VW - WV$ is indeed a vector field, gives its expression in local coordinates, and reveals that

$$[V, W] = L_V W.$$

Review: Why is $\varphi_t \varphi_s = \varphi_{t+s}$? Let M be a smooth manifold, V a smooth vector field on it, and $\{\varphi_t\}$ the associated local flow. Fixing a point $x \epsilon M$,define the curves

$$\alpha(t) = \varphi_t \varphi_s(x) \ \ \textit{and} \ \ \beta(t) = \varphi_{t+s}(x), \ \ \textit{with}$$

$$\alpha(0) = \varphi_s(x) = \beta(0).$$

Then $\alpha'(t) = V(\alpha(t))$ and $\beta'(t) = V(\beta(t))$.

Hence both $\alpha(t)$ and $\beta(t)$ are integral curves of the smooth vector field V with $\alpha(0) = \beta(0)$. By uniqueness of solutions, we have $\alpha(t) = \beta(t)$, and hence $\varphi_t \varphi_s = \varphi_{t+s}$, as desired.

A vector field is invariant under its own flow.

A vector field V on the smooth manifold M is said to be **invariant** under the diffeomorphism h of M if $h_* V = V$.

If $\{\varphi_t\}$ is the local flow of V, we claim that V is invariant under each of the local diffeomorphisms φ_t. Fixing a point x in M, we must show that $(\varphi_t)_* V(x) = V(\phi_t(x))$.

Recall that if V is a tangent vector to M at x, and $f : M \to N$ a smooth map, one of the several definitions of $f_* V$ is to take a smooth curve $\alpha(s)$ in M with $\alpha(0) = x$ and $\alpha'(0) = V$, and then $f_* V = \frac{d}{ds}|_{s=0} f\alpha(s)$.

Now in our case above, we can choose $\alpha(s) = \phi_s(x)$, since $\frac{d}{ds}|_{s=0}\varphi_s(x) = V(x)$. Then, using the fact that $\varphi_t\varphi_s = \varphi_{t+s}$,

$$(\varphi_t)_* V(x) = \frac{d}{ds}|_{s=0}\varphi_t\varphi_s(x)$$

$$= \frac{d}{ds}|_{s=0}\varphi_{s+t}(x) = \frac{d}{ds}|_{s=0}\varphi_s\varphi_t(x)$$

$$= V(\varphi_t(x)).$$

This shows that the vector field V is invariant under its own flow φ_t.

Theorem

A vector field W is invariant under the flow of a vector field V if and only if $L_V W = 0$.

Proof.

Suppose first that W is invariant under the flow φ_t of V:

$$(\varphi_t)_* W(x) = W(\varphi_t(x)).$$

Then

$$(L_V W)(x) =_{defn} lim_{t \to 0}[(\varphi_{-t})_* W(\varphi_t(x)) - W(x)]/t = 0,$$

because the numerator of this difference quotient is identically zero.

Next suppose that $L_V W = 0$. Let us define

$$W(t,x) = (\varphi_{-t})_* W(\varphi_t(x)),$$

so that if x is held fixed and t varies, we have a curve of tangent vectors at x. If we show that this curve has the constant value $W(x)$, then we will have that

$$(\varphi_t)_* W(x) = W(\varphi_t(x)),$$

which will confirm that W is invariant under the flow φ_t of V.

To begin, we write

$$0 = (L_V W)(x) = \frac{d}{dt}|_{t=0}(\varphi_{-t})_* W(\varphi_t(x)) = \frac{d}{dt}|_{t=0} W(t,x).$$

Thus the curve $W(t,x)$ has zero derivative at $t = 0$. We want to show that it has zero derivative for all t, which will imply that it is constant. To that end, we write

$$\begin{aligned}
0 = (L_V W)(\varphi_s(x)) &= \frac{d}{dt}|_{t=0}(\varphi_{-t})_* W(\varphi_t(\varphi_s(x))) \\
&= \frac{d}{dt}|_{t=0}(\varphi_{-t})_* W(\varphi_{t+s}(x)) \\
&= \frac{d}{dt}|_{t=0}(\varphi_s)_*(\varphi_{-t-s})_* W(\varphi_{t+s}(x)) \\
&= \frac{d}{d\tau}|_{\tau=s}(\varphi_s)_*(\varphi_{-\tau})_* W(\varphi_\tau(x)) \\
&= \frac{d}{d\tau}|_{\tau=s}(\varphi_s)_* W(\tau, x) \\
&= (\varphi_s)_* \frac{d}{d\tau}|_{\tau=s} W(\tau, x).
\end{aligned}$$

In the last step above, we used the linearity of $(\varphi_s)_*$ to interchange it with $\frac{d}{d\tau}|_{\tau=0}$, and so learn that

$$(\varphi_s)_* \frac{d}{d\tau}|_{\tau=s} W(\tau, x) = 0.$$

Then, since the linear map $(\varphi_s)_*$ is an isomorphism, we must have $\frac{d}{d\tau}|_{\tau=s} W(\tau, x) = 0$.

This was our goal, since now $W(t, x)$ is constant, with value $W(0, x) = W(x)$, and hence, as indicated above, W is invariant under the flow φ_t of V.

Definition

A real vector space $\mathcal{V} = \{U, V, W, ...\}$ is a *real Lie algebra* if it has a product

$$[\ ,\ ] : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

which is bilinear and satisfies

(1) $[V, V] = 0$, or equivalently, $[V, W] = -[W, V]$

(2) $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$,

known as the **Jacobi identity**.

Thus the space $VF(M)$ of smooth vector fields on a smooth manifold M forms a real Lie algebra.

The subspace of smooth divergence-free vector fields on M forms a Lie subalgebra.

So does the subspace of smooth vector fields on M which are tangent to ∂M.

Example The goal of this example is to compute the Lie algebra of the Lie group S^3 of unit quaternions.

Notational convention. Name a tangent vector to S^3 at the point u by the name of a quaternion orthogonal to u. For example, at the identity 1 , any imaginary quaternion $ai + bj + ck$ will denote a tangent vector there.

(a) Consider the vector fields X, Y and Z on S^3 given by
$X_u = ui, Y_u = uj$, $Z_u = uk$.

Show that the corresponding flows are given by

$$\varphi_t(u) = u(cos(t) + isin(t))$$

$$\psi_t(u) = u(cos(t) + jsin(t)),$$

$$\zeta_t(u) = u(cos(t) + ksin(t)).$$

(b) Compute the Lie bracket $[X, Y]$ directly from the definition,

$$[X, Y]_1 = (L_X Y)_1 = lim_{t\to 0}((((\phi_{-t})_* Y)_1 - Y_1)/t,$$

and show that $[X, Y] = 2Z$. Conclude by symmetry that

$$[Y, Z] = 2X$$

and

$$[Z, X] = 2Y.$$