Problem 1. A particle moves in such a way that its distance from the origin at time t is given by $s(t) = 2\sqrt{t^2+4}$. If $v(t)$ is the velocity of the particle at time t, what is

$$lim_{t\to\infty} v(t)?$$

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{\sqrt{2}}$ (e) 0 (f) ∞

$$S(t) = 2\sqrt{t^2+4}$$

$$S'(t) = V(t) = 2 \cdot \frac{1}{2}(t^2+4)^{-1/2} \cdot (2t)$$

$$= \frac{2t}{(t^2+4)^{\frac{1}{2}}}$$

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} \frac{2t}{\sqrt{t^2+4}}$$

$$= \lim_{t\to\infty} \frac{2t}{\sqrt{t^2+4}} \frac{(1/t)}{(1/t)}$$

$$= \lim_{t\to\infty} \frac{2}{\sqrt{(1/t^2)(t^2+4)}}$$

$$= \lim_{t\to\infty} \frac{2}{\sqrt{1+4/t^2}} = \frac{2}{\sqrt{1+0}} = \boxed{2}$$

Problem 2. What are the global maximum and minimum values of the function
$$f(x) = \frac{x}{1+x^2}?$$

(a) 2 and -2 (b) 1 and -1 (c) 1/2 and -1/2

(d) 2 and 0 (e) 4 and -4 (f) 1 and -1

$$f'(x) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

If $0 = 1-x^2$, then $x = \pm 1$.

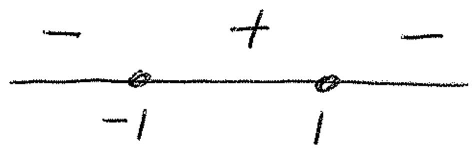
If $0 = (1+x^2)^2$, then there are no solutions.

Hence, the crit ~~po~~ values of $f(x)$ are $x = \pm 1$.

$$f'(0) = \frac{1}{1^2} = 1$$
$$f'(2) = \frac{1-4}{(1+4)^2} = \frac{-3}{25}$$
$$f'(-2) = \frac{1-4}{(1+4)^2} = \frac{-3}{25}$$

Side note: Since $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, then the largest local max is the absolute max and the larges local min is the ab. min.

Hence we have the following schematic for $f'(x)$:



By the first derivative test f has a local max at $x=1$ and a local min at $x=-1$.

Thus, the max. value of f is $\boxed{f(1) = \frac{1}{2}}$

the min value of f is $\boxed{f(-1) = -\frac{1}{2}}$

Problem 3. A stock market analyst sold a monthly newsletter to 320 subscribers at a price of $10 each. She discovered that for each $0.25 increase in the monthly price of the newsletter, she would lose 2 subscriptions. If she sets the price of the newsletters to bring in the greatest total monthly income, what will that income be?

(a) $3200 (b) $4400 (c) $5000

(d) $5800 (e) $6500 (f) $7200

Income = (price of news letter)(# of subscibers)

$I = x \cdot y$

when $x = 10$, $y = 320$.

when $x = 10.25$, $y = 318$

Hence we want to find a formula for the linear relationship between x and y. $y = mx + b$

$$m = \frac{320-318}{10-10.25} = \frac{2}{-1/4} = -8$$

Substitute $x=10$, $y=320$ into $y=-8x+b$

$$320 = -8(10)+b$$

$$b = 400$$

$$I = x(-8x+400) = -8x^2+400x$$

$$I' = -16x+400$$

$$0 = -16x+400$$

$$x = 25$$

Hence, if she charges $25 She will maximize ~~profit~~ Income

To ~~maximi~~ find max income

$$I(25) = (25)(-8(25)+400)$$

$$= 25(200)$$

$$\boxed{= \$5,000}$$

Problem 4. The curve

$$y = x^3 + 3x^2 + ax + b$$

has one inflection point. The tangent line at this inflection point is $y = 3x + 4$. Find the constants a and b.

$y' = 3x^2 + 6x + a$

$y'' = 6x + 6$

To find the inflection point: $0 = 6x + 6$
$x = -1$

Since $y(-1) = (-1)^3 + 3(-1)^2 - a + b = 2 - a + b$

Hence, the inflection point for y is $(-1, 2-a+b)$

To find the slope of the tangent line at $x=-1$:

$y'(-1) = 3(-1)^2 - 6 + a = -3 + a$

Since we know that the tangent line at $(-1, 2-a+b)$ is $y = 3x + 4$, then

$-3 + a = 3 \Rightarrow a = 6$

and $2 - a + b = 3(-1) + 4 \Rightarrow -a + b = -1$

Then substituting $a=6$ into $-a+b=-1$ we get

$-6 + b = -1$

$b = 5$

Hence, $y = x^3 + 3x^2 + 6x + 5$.

Problem 5. A right circular cylinder is inscribed in a cone with height 1 meter and base radius 1 meter. What is the largest possible volume of such a cylinder?

Volume of a cylinder $= \pi r^2 h$

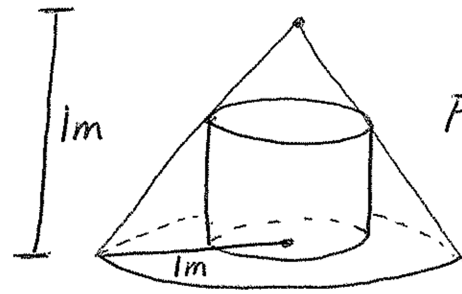


Fig 1

By similar triangles $\frac{1}{1} = \frac{1-h}{r}$.

So $r = 1-h$, or $h = 1-r$

Look at a triangle cross section

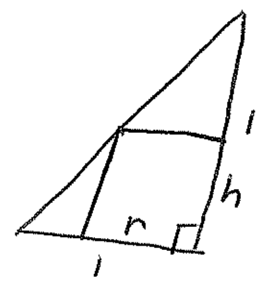


Fig 2

By substitution

$$\cancel{V = \pi(1-h)^2 h}$$

$$V = \pi r^2(1-r)$$

$$V = \pi r^2 - \pi r^3$$

$$V' = 2\pi r - 3\pi r^2$$

Now, check for critical pts

$$0 = 2\pi r - 3\pi r^2$$

$$0 = \pi r(2-3r)$$

$$r = 0 \text{ or } r = \frac{2}{3}$$

So, the largest volume is $V(\frac{2}{3}) = \pi(\frac{2}{3})^2(1-\frac{2}{3})$

$$= \pi \frac{4}{9}(\frac{1}{3})$$

$$= \boxed{\frac{4\pi}{27}}$$

Problem 6. A particle moving on the real line has an acceleration function of $a(t) = cos(t) + sin(t)$. If the particle is at the orgin when $t = 0$ and has a velocity of 5 when $t = 0$, what is the position function for the particle?

$a(t) = \cos(t) + \sin(t) \qquad s(0) = 0,\ V(0) = 5$

Using our formulas for anti-derivatives.

$V(t) = \sin(t) - \cos(t) + a$

$5 = \sin(0) - \cos(0) + a$

$5 = -1 + a$

$a = 6$

So $V(t) = \sin(t) - \cos(t) + 6$

$s(t) = -\cos(t) - \sin(t) + 6t + b$

$0 = -\cos(0) - \sin(0) + 0 + b$

$0 = -1 + b$

$b = 1$

So, $s(t) = -\cos(t) + \sin(t) + 6t + 1$

Problem 7. Let $f(x) = \frac{x^2-4}{x^2+4}$

a) Find the x-intercepts and y-intercepts of $f(x)$.

x-int $(2,0)$ and $(-2,0)$ y-int $(0,-1)$

b) Find the intervals of increase and decrease of $f(x)$.

int. of inc. $= (0, \infty)$ int. of dec. $= (-\infty, 0)$

c) Find the local maxima and local minima of $f(x)$.

local min at $(0,-1)$

d) Find the intervals of concavity of $f(x)$.

c-down $(-\infty,-4) \cup (4/3, \infty)$, cup $(-4, 4/3)$.

e) Find the inflection points of $f(x)$.

$\left(-4, \frac{(-4)^2-4}{(-4)^2+4}\right)$ and $\left(\frac{4}{3}, \frac{\left(\frac{4}{3}\right)^2-4}{\left(\frac{4}{3}\right)^2+4}\right)$

f) Find the horizontal, vertical and slant asymptotes of $f(x)$.

hor. asy. are $\boxed{y=1}$

g) Use all of the above information to **carefully** graph $f(x)$.

a) $0 = \frac{x^2-4}{x^2+4} \Rightarrow x = \pm 2$ so, x-int. are $(2,0)$ and $(-2,0)$

$f(0) = \frac{0^2-4}{0^2+4} = -1$ so, y-int. is $(0,-1)$

b) & c) $f'(x) = \frac{(x^2+4)(2x)-(x^2-4)(2x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}$

Now find crit. pts. $0 = \frac{16x}{(x^2+4)^2} \Rightarrow x = 0$

$f'(1) = \frac{16}{25}$, $f'(-1) = \frac{-16}{25}$

Hence we have the schematic

$-$ $+$

0

d) & e) $f''(x) = \dfrac{(x^2+4)^2(16) - 16x(2(x^2+4)(2x))}{(x^2+4)^4}$

$= \dfrac{16x^4 + 128x^2 + 256 - 64x^4 - 256x^2}{(x^2+4)^4}$

$= \dfrac{-48x^4 - 128x^2 + 256}{(x^2+4)^4}$

Now find crit points

$(x^2+4)^4 \neq 0$ so $f''(x)$ is defined everywhere.

$0 = -48x^4 - 128x^2 + 256$

$0 = -16(3x^4 + 8x^2 - 16)$

$0 = 3x^4 + 8x^2 - 16$

$\cancel{x = \dfrac{-8 \pm \sqrt{64 - 4(3)16}}{2\cdot 3} = \dfrac{-8 \pm \sqrt{-128}}{6}}$

~~Since~~ $\cancel{3x^4 + 8x^2 + 16 > 0}$ ~~then~~ $\cancel{f''(x) < 0}$

~~for all~~ x.

$$x = \frac{-8 \pm \sqrt{64+196}}{6} = \frac{-8 \pm 16}{6} = \tfrac{4}{3} \text{ or } -4$$

$$f''(0) = \frac{256}{4^4}$$

$f''(\text{Big } \#) = \text{negative } \#$

$f''(\text{Big negative } \#) = \text{negative } \#$

So we have the schematic

$$\begin{array}{ccccc} - & & + & & - \\ \hline & -4 & & 4/3 & \end{array}$$

f) no- slant asy.

no vert. asy.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-4}{x^2+4} = \lim_{x\to\infty} \frac{(x^2-4)(1/x^2)}{(x^2+4)(1/x^2)}$$

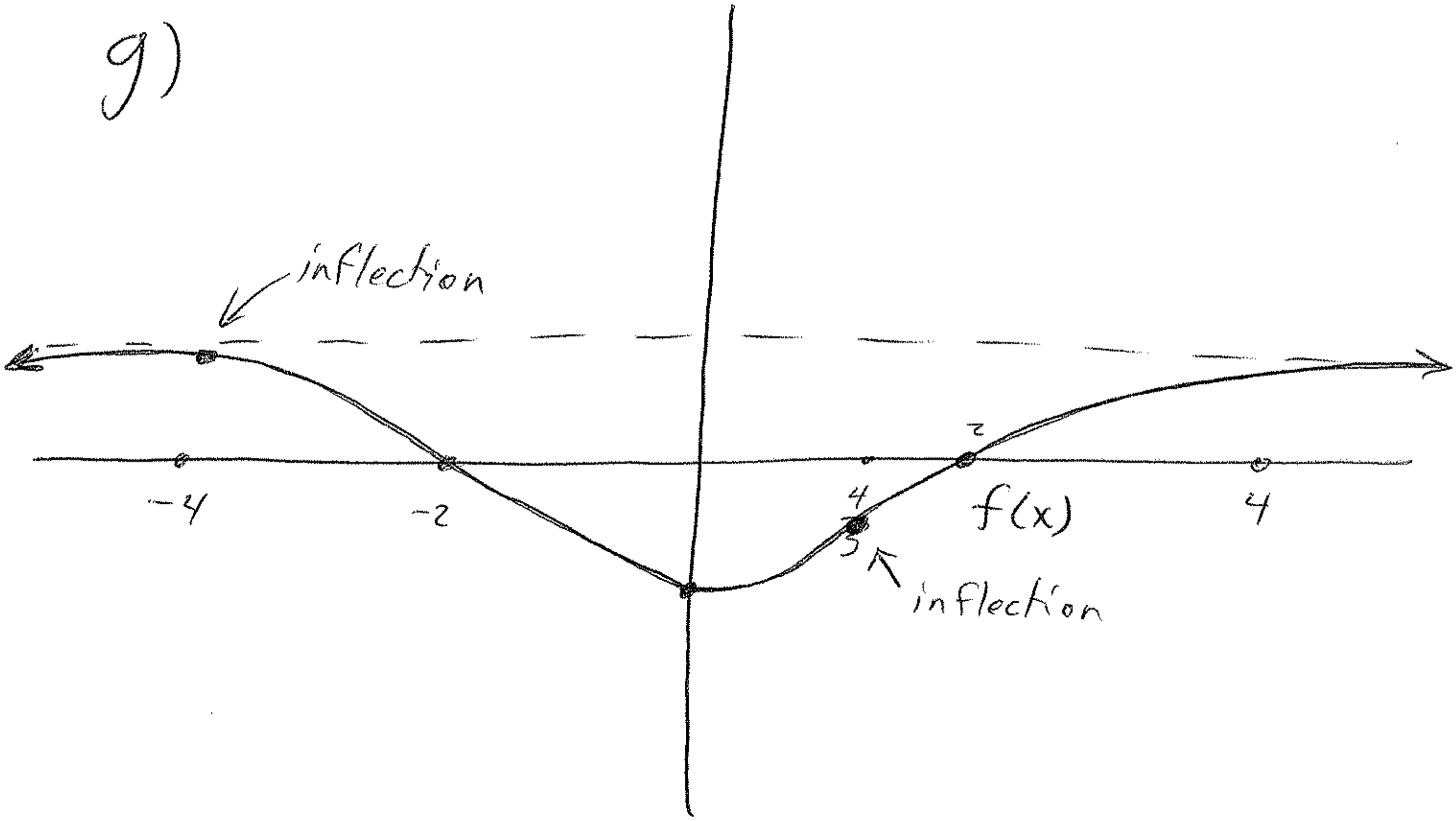
$$= \lim_{x\to\infty} \frac{1 - 4/x^2}{1 + 4/x^2}$$

$$= \frac{1}{1} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-4}{x^2+4} = \lim_{x\to+\infty} \frac{(-x)^2-4}{(-x)^2+4}$$

$$= \lim_{x\to+\infty} \frac{x^2-4}{x^2+4} = 1$$

g)



Problem 8. Find the value of c (if any) that satisfies the conclusion of the Mean Value Theorem for the function $f(x) = \frac{1}{1+x}$ on the interval [0,1] .

a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{\sqrt{2}}{2}$

(d) $2-\sqrt{2}$ (e) $\sqrt{2}-1$ (f) no values

$$f(0) = \frac{1}{1+0} = 1$$

$$f(1) = \frac{1}{1+1} = \frac{1}{2}$$

By MVT there is a value c s.t.

$$f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{\frac{1}{2}-1}{1} = -\frac{1}{2}$$

$$f'(x) = -1(1+x)^{-2} = \frac{-1}{(1+x)^2}$$

Find x when $\frac{-1}{(1+x)^2} = -\frac{1}{2}$

$$2 = (1+x)^2$$

$$\pm\sqrt{2} = 1+x$$

$$-1 \pm \sqrt{2} = x$$

$$\boxed{c = -1+\sqrt{2}}$$