1. PRACTICE MIDTERM 1

Problem 1. At what value(s) of x is the following function discontinuous?

$$f(x) = \begin{cases} x^2+4x+5 & : if \ \ x < -2 \\ \frac{1}{2}x & : if \ \ -2 \le x \le 2 \\ 1+\sqrt{x-2} & : if \ \ x > 2 \end{cases}$$

a) -2
~~b) 0~~
c) -2,0, and 2
d) -2 and 0
e) 2
f) -2 and 2
g) 0 and 2
h) f is continuous everywhere

Since polynomials are cont. in their domain,
$f(x)$ is cont. on $(-\infty, -2)$
and $f(x)$ is cont. on $(-2, 2)$.
Since square root functions are cont. where
their argument is non-negative,
$f(x)$ is cont. on $(2, +\infty)$.
Hence, we need only check continuity at 2 and -2.

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} x^2+4x+5 = (-2)^2+4(-2)+5 = 1$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{1}{2}x = \frac{1}{2}(-2) = -1$$

Hence, $f(x)$ is not cont. at $x=-2$.

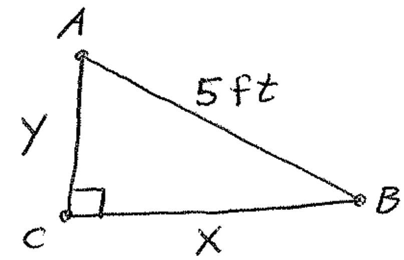
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{2}(x) = \frac{1}{2}(2) = 1$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 1+\sqrt{x-2} = 1+\sqrt{2-2} = 1$$

Since $f(2) = \lim_{x\to 2} f(x)$, then f is cont. at $x=2$.

Problem 2. The hypotenuse AB of a right triangle ABC remains constant at 5 feet as both legs are changing. One leg, AC, is decreasing at the rate of 2 feet per second. In order for the hypotenuse to remain 5 feet, the other leg, BC, is increasing. The rate, in square feet per second, at which the **area** is changing when $AC = 3$ is

a) $\frac{25}{4}$
b) $\frac{7}{2}$
c) $\frac{-3}{2}$
d) $\frac{-7}{4}$
e) $\frac{3}{2}$
f) $\frac{-7}{2}$
g) $\frac{7}{4}$
h) None of these



What we know:

• $\boxed{\frac{dy}{dt} = -2}$ • $x^2 + y^2 = 25$ • $A = \frac{1}{2} x \cdot y$

What we want to know:

what is $\frac{dA}{dt}$ when $\boxed{y = 3}$.

Since $A = \frac{1}{2} xy$, then $\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt} \cdot y + \frac{dy}{dt} x\right)$.

So, to find $\frac{dA}{dt}$ we still need to find $\frac{dx}{dt}$ and x when $y = 3$.

If $y = 3$, then $x^2 + 3^2 = 25$. So, $\boxed{x = 4}$.

Since $x^2 + y^2 = 25$, then $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$.

Plugging in what we know $2(4)\frac{dx}{dt} + 2(3)(-2) = 0$.

Hence, $\boxed{\frac{dx}{dt} = \frac{3}{2}}$

Now we can find $\frac{dA}{dt} = \frac{1}{2}\left(\frac{3}{2}(3) + (-2)(4)\right) = \boxed{\frac{-7}{4} \frac{ft^2}{sec}}$

Problem 3
If $x^2 - xy - y^3 = 13$, then find $\frac{dy}{dx}$ evaluated at $(4, 1)$.

a) 0
b) $\frac{3}{2}$
c) $\frac{7}{2}$
d) $\frac{9}{7}$
e) -2
f) -1
g) 1
h) 7

$$2x - \left(x\frac{dy}{dx} + y\right) - 3y^2\frac{dy}{dx} = 0$$

$$2x - x\frac{dy}{dx} - y - 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(-x - 3y^2\right) = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{-x - 3y^2}$$

$$\frac{dy}{dx} = \frac{1 - 2(4)}{-(4) - 3(1)}$$

$$\frac{dy}{dx} = 1$$

Problem 4. What is the slope of the tangent line to $f(x) = (x)(cos(x^2))$ at $x = \sqrt{\frac{\pi}{2}}$?

a) $-\pi$
b) π
c) 0
d) 1
e) -1
f) $\frac{1}{2}$

First, find $f'(x)$.

$$f'(x) = x \cdot \frac{d}{dx}(\cos(x^2)) + \frac{d}{dx}(x) \cdot \cos(x^2)$$

$$= x \cdot \left(-\sin(x^2) \cdot \frac{d}{dx}(x^2)\right) + \cos(x^2)$$

$$= x \cdot (-\sin(x^2))(2x) + \cos(x^2)$$

$$= -2x^2 \sin(x^2) + \cos(x^2)$$

$$f'\left(\sqrt{\tfrac{\pi}{2}}\right) = -2\left(\sqrt{\tfrac{\pi}{2}}\right)^2 \sin\left(\tfrac{\pi}{2}\right) - \cos\left(\left(\sqrt{\tfrac{\pi}{2}}\right)^2\right)$$

$$= -2\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right) - \cos\left(\tfrac{\pi}{2}\right)$$

$$= -\pi(1) - 0$$

$$= -\pi$$

$$f\left(\sqrt{\tfrac{\pi}{2}}\right) = \sqrt{\tfrac{\pi}{2}} \cos\left(\left(\sqrt{\tfrac{\pi}{2}}\right)^2\right) = \sqrt{\tfrac{\pi}{2}} \cos\left(\tfrac{\pi}{2}\right) = 0$$

Since $f'\left(\sqrt{\frac{\pi}{2}}\right)$ is the slope of the tangent lince and $\left(\sqrt{\frac{\pi}{2}}, 0\right)$ is a point on the line; then, using x-int. form of a line, $\boxed{y = -\pi\left(x - \sqrt{\frac{\pi}{2}}\right)}$ is the tangent line.

Problem 5. The function $f(x) = (x-3)^{\frac{2}{3}}$ is increasing for what values of x?

a) ~~$(-\infty, \infty)$~~
b) $(3, \infty)$
c) ~~nowhere~~
d) $(-\infty, 3)$
e) $(0, \infty)$
f) everywhere except 3

To find where $f(x)$ is increasing, we need to find where $f'(x) > 0$.

$$f'(x) = \frac{2}{3}(x-3)^{\left(\frac{2}{3}-1\right)} \cdot \frac{d}{dx}(x-3)$$

$$= \frac{2}{3}(x-3)^{-1/3} \cdot (1)$$

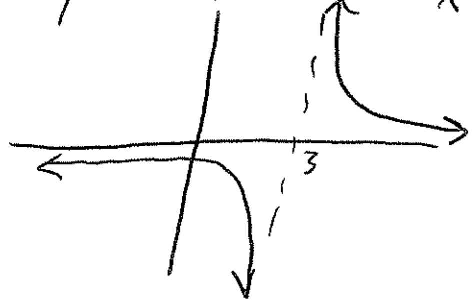
Where is $\frac{2}{3}(x-3)^{-1/3} > 0$?

$$(x-3)^{-1/3} > 0$$

$$\left(\frac{1}{(x-3)^{1/3}}\right)^3 > (0)^3$$

$$\frac{1}{x-3} > 0$$

graph $f(x) = \frac{1}{x-3}$.



Hence $\frac{1}{x-3} > 0$ only when $x > 3$.

Thus, $f'(x) > 0$ only when $x > 3$.

Problem 6.

Use the intermediate value theorem to show that there is a number that is exactly one more than its cube.

We want to find a number a such that

$a = a^3 + 1$.

Let $f(x) = x^3 - x + 1$.

$f(-2) = (-2)^3 + 2 + 1 = -5$

$f(1) = 1^3 - 1 + 1 = 1$

Hence, by the I. V. theorem, there is a number a in the interval $(-2, 1)$ such that $f(a) = 0$.

So $a^3 - a + 1 = 0$

$a = a^3 + 1$.

Thus, we have shown that such an a exists.

Problem 7. Find the value of the limit.

$$lim_{x\to 2}\frac{\sqrt{x+7}-3}{(x-2)(x+1)}$$

$$\lim_{x\to 2}\frac{\sqrt{x+7}-3}{(x-2)(x+1)}=\lim_{x\to 2}\frac{(\sqrt{x+7}-3)(\sqrt{x+7}+3)}{(x-2)(x+1)(\sqrt{x+7}+3)}$$

$$=\lim_{x\to 2}\frac{x+7-9}{(x-2)(x+1)(\sqrt{x+7}+3)}$$

$$=\lim_{x\to 2}\frac{\cancel{x-2}}{\cancel{(x-2)}(x+1)(\sqrt{x+7}+3)}$$

$$=\lim_{x\to 2}\frac{1}{(x+1)(\sqrt{x+7}+3)}$$

$$=\frac{1}{(2+1)(\sqrt{9}+3)}$$

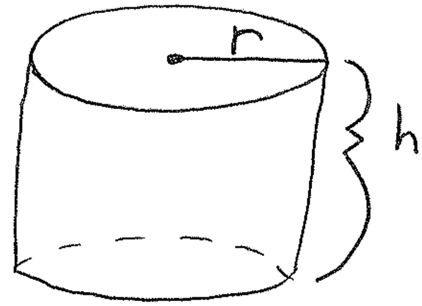
$$=\frac{1}{3(3+3)}$$

$$=\boxed{\frac{1}{24}}$$

Problem 8. Let V be the volume of a cylinder having height h and radius r, and assume that h and r vary with time. When the height is 5 in. and is increasing at 0.2 in./s. , the radius is 3 in. and is decreasing at 0.1 in./s. How fast is the volume changing at that instant?

What we know:

$V = \pi r^2 h$



$\boxed{\frac{dh}{dt} = .2}$

$\boxed{\frac{dr}{dt} = -.1}$

What we want to find:

$\frac{dV}{dt}$ when $\boxed{h = 5}$ and $\boxed{r = 3}$

Since $V = \pi r^2 h$, then $\frac{dV}{dt} = \pi\left(r^2 \cdot \frac{dh}{dt} + h \cdot \frac{d(r^2)}{dt}\right)$

$$\frac{dV}{dt} = \pi\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Now plug in all of the known quantities in boxes.

$$\frac{dV}{dt} = \pi\left(3^2 \cdot (.2) + 2(3)(5)(-.1)\right)$$
$$= \pi(1.8 - 3)$$
$$= \boxed{-1.2\pi \frac{in^3}{sec.}}$$

Problem 9. Suppose $f(3) = 2$, $f'(3) = 5$, and $f"(3) = -2$. Let $g(x) = [f(x)]^2$. Find the value of $g"(3)$.

$$g(x) = (f(x))^2$$

$$g'(x) = 2f(x) \cdot f'(x)$$

$$g''(x) = 2(f(x) \cdot f''(x) + f'(x) \cdot f'(x))$$

$$g''(x) = 2f(x) \cdot f''(x) + 2(f'(x))^2$$

$$g''(3) = 2f(3) \cdot f''(3) + 2(f'(3))^2$$

$$= 2(2)(-2) + 2(5)^2$$

$$= -8 + 50$$

$$= 42$$

$$\boxed{g''(3) = 42}$$

Problem 10. If $f(x) = \frac{x}{tan(x)}$, find $f'(\frac{\pi}{4})$. Do not leave any trigonometric functions in your answer.

$$f(x) = \frac{x}{\tan(x)}$$

$$f'(x) = \frac{\tan(x) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(\tan(x))}{(\tan(x))^2}$$

$$= \frac{\tan(x) - x(\sec(x))^2}{(\tan(x))^2}$$

$$= \frac{\frac{\sin(x)}{\cos(x)} - x\left(\frac{1}{\cos(x)}\right)^2}{\left(\frac{\sin(x)}{\cos(x)}\right)^2}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} - \frac{\pi}{4}\left(\frac{1}{\frac{1}{\sqrt{2}}}\right)^2}{\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)^2}$$

$$= \frac{1 - \frac{\pi}{4}(\sqrt{2})^2}{1}$$

$$= \boxed{1 - \frac{\pi}{2}}$$