Final Exam. Tuesday May 8 from 12:00 - 2:00pm

Meyerson Hall Room B1 (holds 406 people)

Find out where it is before the exam.

Someone came to me and mentioned that they did not see how to do the following problem. (I was impressed because it is hard.)

#11 Fall 15

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1/t & 0 \\ t^2 & 1/t \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2 \\ -t^3 \end{bmatrix}$$

Knowing that $\begin{bmatrix} t \\ t \end{bmatrix}$, $\begin{bmatrix} t \\ 3t \end{bmatrix}$ and $\begin{bmatrix} t+1/t \\ t^2+t \end{bmatrix}$ are solutions

find the general solution.

First seems very hard.

The difference of two solutions to the inhomogeneous is a solution to the homogeneous equation

so $\begin{bmatrix} 0 \\ 2t \end{bmatrix}$ is a solution to the homogeneous equation

It is linear so you can devide by 2 so

$\begin{bmatrix} 0 \\ t \end{bmatrix}$ is a solution to the homogeneous equation

It checks.

Subtracting the first from the third you get

$\begin{bmatrix} 1/t \\ t^2 \end{bmatrix}$ is a solution to the homogeneous equation

It checks.

General solution

$$\frac{d}{dt}\begin{bmatrix} b/t + t \\ at + bt^2 + t \end{bmatrix} =$$

#10 Spring 15

$y'' + 2ky' + 8ky = \cos(2t)$

Find all values of $k \in \mathbb{R}$ so that every solution is bounded as $t \to \infty$. i.e. there is a constant C so that $|y(t)| \leq M$ for $t \geq 0$.

Answer $0 < k < 8$ Wrong.

In general

$Ly = \cos(\omega t) \quad \sin(\omega t) \quad 0$

$y(t)$ is bounded as $t \to \infty$

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

If roots λ are so that $Re(\lambda) < 0$ then bounded $e^{\lambda t}$ bounded

If $Re(\lambda) > 0$ then unbounded $e^{\lambda t}$ unbounded

If $Re(\lambda) = 0$ then further investigation is needed.

$Real(\lambda) = 0$ λ is a simple real root or a simple complex root $\lambda = \pm ib$

Homogeneous equation then it is bounded

Inhomogeneous further investigation

$\lambda = 0$ is a double or triple root then

unbounded $y = A + Bt + Ct^2$

Bt is unbounded

For example

If $y'' + \omega^2 y = A\cos(\omega' t) + B\sin(\omega' t)$

If $\omega \neq \omega'$ bounded. If $\omega = \omega'$

$y = Ct\cos(\omega t) \quad Dt\sin(\omega t)$ unbounded

$Re(\lambda) > 0$ unbounded as $t \to \infty$

$Re(\lambda) < 0$ bounded as $t \to \infty$

$Re(\lambda) = 0$ further investigation

Return to the original problem

$y'' + 2ky' + 8ky = \cos(2t)$

Find all values of $k \in \mathbb{R}$ so that every solution is bounded as $t \to \infty$. i.e. there is a constant C so that $|y(t)| \leq M$ for $t \geq 0$.

Homogeneous equation

$$\lambda^2 + 2k\lambda + 8k = 0$$

$$\lambda = \frac{-2k \pm \sqrt{4k^2-32k}}{2} = -k \pm \sqrt{k^2-8k}$$

If $k = 0$ $\lambda^2 = 0$ $y(t) = A + Bt + (C_1\cos(2t) + C_2\sin(2t))$

If $k < 0$ positive roots so $Ce^{\lambda t}$ not bounded

Then $-k + \sqrt{k^2-8k}$ is positive. Has positive root so unbound.

If $k = 0$ $\lambda^2 = 0$ so double root $A + Bt$ unbounded

If $0 < k < 8$ complex roots $a \pm ib$ $a < 0$

$e^{at}(\cos + \sin)$ $a < 0$ so bounded

the $\cos(2t)$ term just give $A\cos(2t) + B\sin(2t)$

$k = 8$ $\lambda = -8$ double root bounded $t^n e^{-ct} \to 0$ $t \to \infty$

$k > 8$ two real roots both negative

Answer $0 < k < 8$ is WRONG

Correct answer. $0 < k$ correct.

$Ly = r(x)e^{\lambda x}$ $r(x)$ a polynomial $Ly = P(D)y$

$p(\lambda)$ is called the auxilary polynomial.

If λ is not a root of the auxilary polynomial

$y_p(x) = q(x)e^{\lambda x}$ that is the same order as r

If λ is a simple root

$y_p(x) = q(x)x\, e^{\lambda x}$ q is the same order as r

If λ is a double root

$y_p(x) = q(x)x^2\, e^{\lambda x}$

#9 Spring 15

$$\left(\frac{d}{dx} + 1\right)^3\left(\frac{d}{dx} - 1\right)y = -240x^2e^{-x} + 120e^{-x}$$

$$\left(\frac{d}{dx} + 1\right)^2 = \frac{d^2}{dx^2} + 2\frac{d}{dx} + 1$$

$$\left(\frac{d}{dx} + 1\right)^3 = \frac{d^3}{dx^3} + 3\frac{d^2}{dx^2} + 3\frac{d}{dx} + 1$$

$$\left(\frac{d}{dx} + 1\right)^3\left(\frac{d}{dx} - 1\right) = \left(\frac{d^3}{dx^3} + 3\frac{d^2}{dx^2} + 3\frac{d}{dx} + 1\right)\left(\frac{d}{dx} -1\right)$$

$$= \frac{d^4}{dx^4} + 2\frac{d^3}{dx^3} - 2\frac{d}{dx} - 1$$

Particular solution is $y_p(x) = q(x)e^{-x}$

trick

$$\left(\frac{d}{dx} + 1\right) q(x)e^{-x} = q'e^{-x} - qe^{-x} + qe^{-x} = q'e^{-x}$$

$$Ly = \left(\frac{d}{dx} + 1\right)^3\left(\frac{d}{dx} - 1\right)qe^{-x} = \left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} + 1\right)^3qe^{-x} = \left(\frac{d}{dx} - 1\right)q'''(x)e^{-x}$$

$Lqe^{-x} = q''''e^{-x} - q'''e^{-x} - q'''e^{-x} = (q'''' - 2q''')e^{-x} = -240x^2e^{-x} + 120e^{-x}$

multiplying by e^x we get

$$q'''' - 2q''' = -240x^2 + 120$$

let $u = q'''$

$$u' - 2u = -240x^2 + 120$$

use the formula for first order differential equation.

You can but you know the answer is a polynomial

$$u = ax^2 + bx + c.$$

highest power is two

$$-2a = -240 \quad \text{so } a = 120$$

so $u = 120x^2 + bx + c$

$$u' = 240x + b$$

$$u' - 2u = 240x + b - 240x^2 - 2bx - 2c = -240x^2 + 120$$

The x^2 terms cancel and we get

$$240x - 2bx + b-2c = 120$$

$$b = 120$$

$$c = 0$$

the answer is

$$u = 120x^2 + 120x$$

$$q''' = u$$

So to get u we integrate 3 times.

one integration gives

$40x^3 + 60x^2 + C_3$

two integrations gives

$10x^4 + 20x^3 + C_3x + C_2$

three integrations gives

$2x^5 + 5x^4 + \frac{1}{2}C_3x^2 + C_2x + C_1$

replace $\frac{1}{2}C_3$ by C_3 (just the name of the constant)

General solution

$$y(x) = (2x^5 + 5x^4 + C_3x^2 + C_2x + C_1)e^{-x} + C_4e^x$$