9.9 Phase Plane for Autonomous System

$$\frac{dx}{dt} = F(x,y)$$

$$\frac{dy}{dt} = G(x,y)$$

First find the equilibrium points. These are points (x_0,y_0) so that $F(x_0,y_0) = 0$ and $G(x_0,y_0) = 0$.

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$$\frac{dx}{dt} = x^2 - y = F(x,y)$$

$$\frac{dy}{dt} = x - y = G(x,y)$$

Equilibrium point $x^2 = y$ $x = y$ so $x^2 = x$ so

$x = 0, 1$ Equilibrium points $(0,0)$ and $(1,1)$

At an equilibrium point (x_0,y_0) you have the solution

$x(t) = x_0$ $y(t) = y_0$.

You want to analyze the equilibrium point. Is it stable etc.

We want to analyze the equilibrium point (x_0,y_0).

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{New variables} \quad \vec{x}(t) = \begin{bmatrix} x_0 + x(t) \\ y_0 + y(t) \end{bmatrix}$$

so now $x(t)$, $y(t)$ represent the displacement of $\vec{x}(t)$ from the equilibrium point (x_0,y_0).

Now your equations of motion are

$$\frac{dx}{dt} = F(x+x_0, y+y_0)$$

$$\frac{dy}{dt} = G(x+x_0, y+y_0)$$

and $(x(t),y(t)) = (0,0)$ is a solution.

Now we linearize the problem If F is differentiable at (x_0,y_0)

$$F(x+x_0,y+y_0) = F(x_0,y_0) + x\,\frac{\partial F}{\partial x}(x_0,y_0) + y\,\frac{\partial F}{\partial y}(x_0,y_0) + o(\sqrt{x^2+y^2})$$

error $o(x,y) = \sqrt{x^2+y^2}\; e(x,y)$ $e(x,y) \longrightarrow 0$ as $\sqrt{x^2+y^2} \longrightarrow 0$

We replace $F(x+x_0,y+y_0)$ by

$$F(x+x_0,y+y_0) \longrightarrow ax + by \quad \text{with} \quad a = \frac{\partial F}{\partial x}(x_0,y_0) \quad b = \frac{\partial F}{\partial y}(x_0,y_0)$$

Likewise we replace $G(x+x_0,y+y_0)$ with

$$G(x+x_0,y+y_0) \longrightarrow cx + dy \quad \text{with} \quad c = \frac{\partial G}{\partial x}(x_0,y_0) \quad d = \frac{\partial G}{\partial y}(x_0,y_0)$$

And now we are back the linear problem

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

Eigenvalues	Type of Equilibrium Point
Real and negative	Stable node
Real and positive	Unstable node
Opposite sign	Saddle
Complex with positive real part	Unstable spiral
Complex with negative real part	Stable spiral
Pure imaginary	Stable center (linear system) Center or spiral point stability indeterminate

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$$\frac{dx}{dt} = x^2 - y = F$$

$$\frac{dy}{dt} = x - y = G$$

Find and classify the equilibrium points

$x^2 = y \quad x = y \qquad x^2 = x \quad$ so $x = 0,1$

Equilibrium points $(0,0)$ $(1,1)$

$$\frac{\partial F}{\partial x} = 2x \qquad \frac{\partial F}{\partial y} = -1$$

$$\frac{\partial G}{\partial x} = 1 \qquad \frac{\partial G}{\partial y} = -1$$

$$(0,0) \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$\mathrm{tr}(A) = -1 \qquad \det(A) = 1$

$$\lambda^2 + \lambda + 1 = 0 \qquad \lambda = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

$\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$

stable spiral

$(1,1)$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

$\mathrm{tr}(A) = 1 \qquad \det(A) = -2+1 = -1$

$\lambda^2 + \lambda - 1 = 9$

$$\lambda = \frac{-1 \pm \sqrt{1+4}}{2} = \tfrac{1}{2}(\sqrt{5}-1) \quad -\tfrac{1}{2}(\sqrt{5} + 1)$$

positive negative alternate sign

Saddle point.

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$$\frac{dx}{dt} = y(t) \qquad \frac{dy}{dt} = -4\sin(x(t)) - \frac{y(t)}{25}$$

Equilibrium points

$$y = 0 \qquad -4\sin(x) - y/25 = 0$$

$$y = 0 \qquad \sin(x) = 0 \qquad x = n\pi \quad y = 0$$

$$a = \frac{\partial F}{\partial x}(x_0, y_0) \qquad b = \frac{\partial F}{\partial y}(x_0, y_0)$$

$$c = \frac{\partial G}{\partial x}(x_0, y_0) \qquad d = \frac{\partial G}{\partial y}(x_0, y_0)$$

$F(x,y) = y$ so $a = 0$ $b = 1$

$G(x,y) = -4\sin(x) - y/25$

$c = -4\cos(x)$ $d = -1/25$

$x = n\pi$

$c = -4$ n even $c = 4$ n odd

n even $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & \frac{-1}{25} \end{bmatrix}$$

$$\lambda^2 + \frac{\lambda}{25} + 4 = 0$$

$$\lambda = \frac{-0.04 \pm \sqrt{0.0016 - 16}}{2}$$

stable spiral

n odd $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & \frac{-1}{25} \end{bmatrix}$$

$$\lambda^2 + \frac{\lambda}{25} - 4 = 0$$

$$\lambda = \frac{-0.04 \pm \sqrt{0.0016 + 16}}{2}$$

Saddle

Remark Systems of differential equations

$$\frac{dx}{dt} = 2x - y + 7z \quad \frac{dy}{dt} = x - y + 3z \quad \frac{dz}{dt} = -x + y - z$$

Same as

$$\frac{d\vec{r}}{dt}(t) = \begin{bmatrix} 2 & -1 & 7 \\ 1 & -1 & 3 \\ -1 & 1 & -1 \end{bmatrix} \vec{r}(t)$$