Reduction of order. (This is important.)

$$y'' + a_1(x)y' + a_2(x)y = F(x)$$

Suppose you know a solution y_1 to the homogeneous equation so

$$y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0.$$

Try $y(x) = u(x)y_1(x) = uy_1$

Try $y(x) = u(x)y_1(x) = uy_1$

$$y' = u'y_1 + uy_1'$$

$$y'' = u''y_1 + 2u'y_1' + uy_1''$$

plug in to equation you want to solve.

$$u''y_1 + 2u'y_1' + uy_1'' + a_1u'y_1 + a_1uy_1' + a_2uy_1 = F$$

Look at the terms with u (as apposed to u′ or u″). They are

$$u(y_1'' + a_1y_1' + a_2y_1) = u(0) = 0$$

so you are left with

$$u''y_1 + 2u'y_1' + a_1u'y_1 = F$$

This is a first order differential equation in u'. Let $w = u'$

$$w'y_1 + 2wy_1' + a_1wy_1 = F$$

or

$$w' + (2y_1'/y_1)w + a_1w = F/y_1$$

Write this down solution to linear first order differential equation.

$$y'(x) + p(x)y(x) = q(x)$$

$$y(x) = e^{-\int p(x)\,dx}\left[\int q(x)\,e^{\int p(x)\,dx}\,dx + C\right]$$

#7 Fall 15

$$y'' - 6y' + 9y = e^{3x}\ln(x)$$

using $y = e^{3x}$ is a solution to the homogeneous equation.

$p(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$

Homogeneous equation $C_1e^{3x} + C_2x\, e^{3x}$

$y = ue^{3x}$

$$y' = u'e^{3x} + 3ue^{3x}$$

$$y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

$Ly = u''e^{3x} + 6u'e^{3x} + 9ue^{3x} - 6u'e^{3x} - 18ue^{3x} + 9ue^{3x} = \ln(x)e^{3x}$

Note u terms cancel

$$u''\, e^{3x} = \ln(x)\, e^{3x}$$

$$u'' = \ln(x)$$

Put on your page of notes

$$\int x^n \ln(x)\, dx = \frac{x^{n+1}}{n+1}\ln(x) - \frac{x^{n+1}}{(n+1)^2}$$

$u' = x\ln(x) - x + c_2$

$u = ½x^2\ln(x) - (1/4)\, x^2 - ½x^2 + c_2x + c_1$

$y = ((½\, x^2\ln(x) - (3/4)x^2) + C_1 + C_2x)e^{3x}$

#10 Fall 16

$$4x^2y'' + y = 24\sqrt{x}\ \ln(x)$$

Indicial equation $4(\lambda(\lambda-1)) + 1 = 0$

$$4\lambda^2 - 4\lambda + 1 = 0$$

$4(\lambda^2-\lambda + 1/4) = 0$ so $4(\lambda-\frac{1}{2})^2 = 0$ $\lambda = \frac{1}{2}$ double root

Homogeneous equation $y = C_1x^{\frac{1}{2}} + C_2x^{\frac{1}{2}}\ln(x)$

Try $y = ux^{\frac{1}{2}}$

$$y' = u'x^{\frac{1}{2}} + \tfrac{1}{2}ux^{-\frac{1}{2}}$$

$$y'' = u''x^{\frac{1}{2}} + \tfrac{1}{2}u'x^{-\frac{1}{2}} + \tfrac{1}{2}u'x^{-\frac{1}{2}} - (1/4)ux^{-3/2}$$

$$4x^2y'' + y = 4u''x^{5/2} + 4u'x^{3/2} - ux^{\frac{1}{2}} + ux^{\frac{1}{2}} = 24x^{\frac{1}{2}}\ln(x)$$

divide by $4x^{5/2}$

$$u'' + (1/x)u' = 6x^{-2}\ln(x)$$

recall $y' + py = q$

$$y(x) = e^{-\int p(x)\,dx}\left[\int q(x)\, e^{\int p(x)\,dx}\,dx + C\right]$$

$p(x) = 1/x$ $\int 1/x\,dx = \ln(x)$ $e^{-\ln(x)} = 1/x$

$$e^{\ln(x)} = x$$

$$u' = x^{-1}\left[\int x\ 6x^{-2}\ln(x)\,dx + C\right]$$

Need to do $\int 6/x\ \ln(x)\,dx$

page of notes $\displaystyle\int \frac{\ln^n(x)}{x}\,dx = \frac{\ln^{n+1}(x)}{n+1}$

so

$$u' = x^{-1}((6/2)\ \ln^2(x) + C_2)$$

$$u' = \frac{3\ln^2(x)}{x} + C_2/x$$

integrating again we find

$$u = \ln^3(x) + C_2\ln(x) + C_1$$

so $y = x^{\frac{1}{2}}u = y = \sqrt{x}(\ln^2(x) + C_1 + C_2\ln(x))$

We know C_1 and C_2 terms a solution to the homogeneous equation.

Set $C_1 = C_2 = 0$

$$y = x^{\frac{1}{2}}\ln^3(x)$$

$$y' = \tfrac{1}{2}x^{-\frac{1}{2}}\ln^3(x) + 3x^{-\frac{1}{2}}\ln^2(x)$$

$$y'' = -(1/4)\ x^{-3/2}\ln^3(x) + 3/2\ x^{-3/2}\ln^2(x) - 3/2\ x^{-3/2}\ln^2(x) + 6x^{-3/2}\ln(x)$$

$$4x^2y'' = -x^{\frac{1}{2}}\ln^3(x) + 24x^{\frac{1}{2}}\ln(x)$$

$$4x^2y'' + y = 24\ x^{\frac{1}{2}}\ln(x)$$

Another example.

$y'' + (4x+1)y' + (4x^2+2x+2)y = 0 \qquad y = Ce^{-x^2}$

$$y = ue^{-x^2}$$

$$y' = (u'-2xu)e^{-x^2}$$

$$y'' = (u'' - 2xu' - 2u - 2xu' + 4x^2u)e^{-x^2}$$

collect u" only one

collect u' terms   note u terms cancel

$$u'' - 4xu' + 4xu' + u' \quad \text{so } u'' + u' = 0$$

$$u' = Ce^{-x} \qquad u = Ce^{-x}$$

$$y = C_1 e^{-x^2} + C_2 e^{-x} e^{-x^2}$$

Next page is some notes for your page of notes = 3½ inches < 1/3 page

$$\int x^n \ln(x)\,dx = \frac{x^{n+1}}{n+1}\ln(x) - \frac{x^{n+1}}{(n+1)^2} \qquad \int \frac{\ln^n(x)}{x}\,dx = \frac{\ln^{n+1}(x)}{n+1}$$

$$\int x^n \ln^2(x)\,dx = \frac{x^{n+1}}{n+1}\ln^2(x) - \frac{2x^{n+1}}{(n+1)^2}\ln(x) + \frac{2x^{n+1}}{(n+1)^3}$$

$$\int x\sin(ax)dx = -(x/a)\cos(ax) + a^{-2}\sin(ax) \qquad \int x\cos(ax)dx = (x/a)\sin(ax) + a^{-2}\cos(ax)$$

$$\int \tan(ax)\,dx = -a^{-1}\ln(\cos(ax)) \qquad \int \cot(ax)\,dx = a^{-1}\ln(\sin(ax))$$

$$\int \sec(ax)dx = \int \frac{dx}{\cos(ax)} = a^{-1}\ln\left(\frac{1 + \sin(ax)}{\cos(ax)}\right)$$

$$\int \csc(ax)\,dx = \int \frac{dx}{\sin(ax)} = -a^{-1}\ln\left(\frac{1 + \cos(ax)}{\sin(ax)}\right)$$

9.9 Phase Plane for a Linear Autonomous system

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad \text{solution is } \vec{x}(t) = e^{tA}\vec{x}(0)$$

$\vec{x}(t)$ is a two dimensional vector function of t.

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

What do solutions look like.

We look at the paths (x(t),y(t))

curve (x(t),y(t))

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{cx + dy}{ax + by}$$

Look at eigenvalues

$$\lambda^2 - tr(A)\lambda + det(A) = 0$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\lambda = \frac{(a+d)\pm\sqrt{a^2+2ad+d^2-4ad+4bc}}{2}$$

$$\lambda = \frac{(a+d)\pm\sqrt{(a-d)^2+4bc}}{2}$$

Eigenvalues	Type of Equilibrium Point
Real and negative	Stable node
Real and positive	Unstable node
Opposite sign	Saddle
Pure imaginary	Stable center
Complex with positive real part	Unstable spiral
Complex with negative real part	Stable spiral

Negative eigenvalues flow in

Positive eigenvalues flow out

Non linear autonomous systems (next lecture)