Cauchy-Euler equation.

$$Ly = x^n\frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0$$

$$x\frac{dx^r}{dx} = rx^r$$

$$x^2\frac{d^2}{dx^2}x^r = r(r-1)x^r$$

$$x^3\frac{d^3}{dx^3}x^r = r(r-1)(r-2)x^r$$

$$x^k\frac{d^k}{dx^k}(x^r) = r(r-1)(r-2)\cdots(r-k+1)x^r$$

So $Lx^r = p(r)x^r$ p = indicial polynomial

$p(r) = 0$ indicial equation

e.g. $$3x^3\frac{d^3y}{dx^3} + 5x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$$

$$p(r) = 3r(r-1)(r-2) + 5r(r-1) + 2r + 7$$

$$= 3(r^3 - 3r^2 + 2r) + 5(r^2-r) + 2r - 2$$

$$= 3r^3 - 4r^2 + 3r - 2$$ Note $r = 1$ is a root.

$$= (r - 1)(3r^2 - r + 2)$$

$$r = \frac{1 \pm \sqrt{1-24}}{6}$$

Find the roots $r_1, r_2, \cdots, r_m$

for simple root r_i Cx^{r_i}

double root r_i $C_1x^{r_i} + C_2x^{r_i}\ln(x)$

triple root r_i $C_1x^{r_i} + C_2x^{r_i}\ln(x) + C_3x^{r_i}\ln^2(x)$

and so on.

complex roots (occur in pairs $r = a \pm ib$)

$$x^{a+ib} = x^a x^{ib} = x^a e^{ib\cdot\ln(x)} = x^a(\cos(b\cdot\ln(x)) + i \sin(b\cdot\ln(x)))$$

$$C_1x^a \cos(b\cdot\ln(x)) + C_2 \sin(b\cdot\ln(x))$$

Double complex root

$$C_1x^a \cos(b\cdot\ln(x)) + C_2 \sin(b\cdot\ln(x))$$

$$C_3x^a \ln(x)\cos(b\cdot\ln(x)) + C_4 \ln(x)\sin(b\cdot\ln(x))$$

#9 Fall 16

Find general solution to $x^2y'' + xy' + 36y = 0$

Indicial equation $\lambda(\lambda-1) + \lambda + 36 = 0$

$$\lambda^2 + 36 = 0$$

$$\lambda = \pm 6i$$

$$y = C_1\cos(6\ln(x)) + C_2\sin(6\ln(x))$$

Variation of parameters. (skip)

Reduction of order. (This is important.)

I have used this method many times myself.

$$y'' + a_1(x)y' + a_2(x)y = F(x)$$

Suppose you know a solution y_1 to the homogeneous equation so

$$y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0.$$

Try $y(x) = u(x)y_1(x) = uy_1$

Try $y(x) = u(x)y_1(x) = uy_1$

$$y' = u'y_1 + uy_1'$$

$$y'' = u''y_1 + 2u'y_1' + uy_1''$$

plug in to equation you want to solve.

$$u''y_1 + 2u'y_1' + uy_1'' + a_1u'y_1 + a_1uy_1' + a_2uy_1 = F$$

Look at the terms with u (as apposed to u′ or u″). They are

$$u(y_1'' + a_1y_1' + a_2y_1) = u(0) = 0$$

so you are left with

$$u''y_1 + 2u'y_1' + a_1u'y_1 = F$$

This is a first order differential equation in u′. Let w = u′

$$w'y_1 + 2wy_1' + a_1wy_1 = F$$

or

$$w' + (2y_1'/y_1)w + a_1w = F/y_1$$

Write this down solution to linear first order differential equation.

$$y'(x) + p(x)y(x) = q(x)$$

$$y(x) = e^{-\int p(x)\,dx} \left[\int q(x)\, e^{\int p(x)\,dx}\, dx + C \right]$$

#7 Fall 15

$$y'' - 6y' + 9y = e^{3x} \ln(x)$$

using $y = e^{3x}$ is a solution to the homogeneous equation.

$$p(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

Homogeneous equation $C_1 e^{3x} + C_2 x\, e^{3x}$

$y = ue^{3x}$

$$y' = u'e^{3x} + 3ue^{3x}$$

$$y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

$$Ly = u''e^{3x} + 6u'e^{3x} + 9ue^{3x} - 6u'e^{3x} - 18ue^{3x} + 9ue^{3x} = \ln(x)e^{3x}$$

Note u terms cancel

$$u''\, e^{3x} = \ln(x)\, e^{3x}$$

$$u'' = \ln(x)$$

Put on your page of notes

$$\int x^n \ln(x)\, dx = \frac{x^{n+1}}{n+1} \ln(x) - \frac{x^{n+1}}{(n+1)^2}$$

(include this to be safe) $$\int x^n \ln^2(x)\, dx = \frac{x^{n+1}}{n+1} \ln^2(x) - \frac{2x^{n+1}}{(n+1)^2} \ln(x) + \frac{2x^{n+1}}{(n+1)^3}$$

$$u' = x \ln(x) - x + c_2$$

$$u = \tfrac{1}{2}x^2 \ln(x) - (1/4)\, x^2 - \tfrac{1}{2}x^2 + c_2 x + c_1$$

$$y = ((\tfrac{1}{2}\, x^2 \ln(x) - (3/4)x^2) + C_1 + C_2 x)e^{3x}$$