Last time. The harmonic oscillator equation.

$$m \frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

m = mass   k = spring constant   c = damping constant.

Indicial equation

$$m\lambda^2 + c\lambda + k = 0$$

$$\lambda = \frac{-c \pm \sqrt{c^2-4km}}{2m}$$

Underdamped. $c^2-4km < 0$ (typical as damping is small)

Let $\mu = \sqrt{4km-c^2}$

Solution is

$$y(t) = Ae^{-ct/2m} \cos(\mu t) + Be^{-ct/2m} \sin(\mu t)$$

$$y(t) = Ae^{-ct/2m} \cos(\mu t+\varphi)$$ draw picture.

Critically damped case $c^2 - 4km = 0$

double root $\lambda = -c/2m$

Solution is

$$y(t) = (A+Bt)e^{-ct/2m}$$

Take the case where $A > 0$ so $y(0) = A > 0$

If $B \geq 0$ system does not pass through equilibrium.

draw picture

Maximum occurs at when $y'(t) = 0$ at

$$y'(t) = -c(A+Bt)/2m \; e^{-ct/2m} + B\, e^{-ct/2m}$$

$y'(t) = 0$ when $2mB = cA + cBt$

$$t = (2mB-cA)/cB = 2m/c - A/B$$

if t negative then maximum at $t = 0$. draw picture.

if $A > 0$ and $B < 0$ then system passes through equilibrium when $(A+Bt) = 0$ or $t = -A/B$

Over damped.

$$c^2 - 4km > 0 \qquad \mu = \sqrt{c^2-4km}$$

Two roots

$$\lambda = \frac{-c \pm \mu}{2m} \qquad \text{note } \mu < c \text{ so both roots negative}$$

call two roots $-\lambda_1$ and $-\lambda_2$

$$y(t) = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t}$$

Forced oscillations

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = K\cos(\omega t)$$

if $c > 0$ $\qquad A\cos(\omega t) + B\sin(\omega t)$

solve for A and B $\quad$ The amplitude $= \sqrt{A^2+B^2}$

$$\frac{dy}{dt} = \omega\,(-A\sin(\omega t) + B\cos(\omega t))$$

The amplitude of $\frac{dy}{dt}$ is $\omega\sqrt{A^2+B^2}$

We want to compute the amplitude of $\frac{dy}{dt}$ $\quad$ Kenetic energy is $\frac{1}{2}mv^2 = \frac{1}{2}m(\frac{dy}{dt})^2$

We will use the complex solution.

$$y = (A+iB)e^{i\omega t} \qquad \text{take the real part.}$$

$$\text{amplitude } |A+iB| = \sqrt{A^2+B^2}$$

$$y = Ae^{i\omega t}$$

$$A(-m\omega^2 + ic\omega + k)e^{i\omega t} = K\,e^{i\omega t}$$

$$A(-m\omega^2 + ic\omega + k) = K$$

$$A((k-m\omega^2)+ic\omega) = K$$

$$A((k-m\omega^2)^2 + c^2\omega^2) = ((k-m\omega^2)-ic\omega)K$$

$$A = \frac{((k-m\omega^2)-ic\omega)K}{(k-m\omega^2)^2 + c^2\omega^2}$$

$$\text{Amplitude} = ((k-m\omega^2)^2 + c^2\omega^2)^{-1/2} K$$

Amplitude of the velocity $\frac{dy}{dt}$ $= \omega$ (Amplitude of y)

$$\omega A = (\omega^{-2}(k-m\omega^2)^2 + c^2)^{-1/2}$$

Maximum occurs when $k = m\omega^2$ or $\omega = \omega_0 = \sqrt{k/m}$

ω_0 = angular frequency if $c = 0$

$\omega A = ((m/\omega)^2(\omega^2-\omega_0^2)^2 + c^2)^{-1/2}$ draw picture.

Scale things so max $\omega A = 1$ and $\omega_0 = 1$

$x \leftrightarrow \omega$ scaled $y \leftrightarrow \omega A(\omega)$ so $\omega = x\omega_0$ $y = cA_0(x\omega_0)$

$$y(x) = c((m/(x\omega_0))^2(x^2\omega_0^2-\omega_0^2)^2 + c^2)^{-1/2}$$

$$y(x) = c((m\omega_0/x)^2(x^2-1)^2 + c^2)^{-1/2}$$

$$y(x) = ((m\omega_0/c)^2(x-1/x)^2 + 1)^{-1/2}$$

$$y(x) = \frac{1}{\sqrt{(Q^2(x-1/x)^2 + 1)}}$$

Q = the Q of the oscillator. $Q = m\omega_0/c = m\sqrt{k/m}\,/c = \frac{\sqrt{mk}}{c}$

Q is big sharp resonance

Am radio freqencies 540-1600 kHz kilo Hertz

540,000 to 1,600,000 cycles per second

10 kHz between stations 540 kHz 550 kHz 10/540 = 1/54

You don’t want cross talk so Q has to be much bigger than 54.

I could not find actual values for Q but clearly you need Q’s in the thousands.

Cauchy-Euler equation.

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdot + a_{n-1}\frac{dy}{dx} + a_ny = 0$$

x^r

$$r(r-1)\cdots\cdot(r-n+1) + a_1r(r-1)\cdots(r-n+2) + a_{n-1}r + a_n = 0$$

collect powers of r

$$r^n + b_1r^{n-1} + \cdots\cdot + b_{n-1}r + a_n = 0$$

Find the roots r_1, r_2, ····, r_m

for simple root r_i $\quad Cx^{r_i}$

double root r_i $\quad C_1x^{r_i} + C_2x^{r_i}\ln(x)$

triple root r_i $\quad C_1x^{r_i} + C_2x^{r_i}\ln(x) + C_3x^{r_i}\ln^2(x)$

and so on.

complex roots (occur in pairs $r = a \pm ib$)

$$x^{a+ib} = x^a\, x^{ib} = x^a\, e^{ib\cdot\ln(x)} = x^a(\cos(b\cdot\ln(x)) + i\,\sin(b\cdot\ln(x)))$$

$$C_1x^a\cos(b\cdot\ln(x)) + C_2\,\sin(b\cdot\ln(x))$$

Double complex root

$$C_1x^a\cos(b\cdot\ln(x)) + C_2\,\sin(b\cdot\ln(x))$$

$$C_3x^a\ln(x)\cos(b\cdot\ln(x)) + C_4\,\ln(x)\sin(b\cdot\ln(x))$$