#11 Fall 2016

Find the general solution to $y'' + 2y' + 5y = 8e^{-x}\cos(2x)$

Too hard for now   try $y'' + 2y' + 5y = 8e^{x}\cos(2x)$

Find the general solution to the homogeneous equation

$\lambda^2 + 2\lambda + 5 = 0$ $\lambda = \dfrac{-2\pm\sqrt{4-4\cdot 5}}{2} = -1 \pm \sqrt{1-5} = -1 \pm 2i$

$C_1e^{-x}\cos(2x) + C_2e^{-x}\sin(2x)$

Try $Ae^{x}\cos(2x) + Be^{x}\sin(2x)$

plug in solve for A and B.

Complex method

$$y'' + 2y' + 5y = 8e^{x}e^{2ix} = 8e^{x}(\cos(2x) + i\sin(2x))$$

take the real part

$y = Ae^{(1+2i)x}$

$y' = A(1+2i)e^{(1+2i)x}$

$y''(x) = A(1+2i)^2e^{(1+2i)x} = A(-3+4i)e^{(1+2i)x}$

equation becomes

$$A((-3+4i) + 2 + 4i + 5)e^{(1+2i)x} = 8e^{(1+2i)x}$$

$A(4+8i) = 8$

$A(4+8i)(4-8i) = 32 - 64i$

$A(16+64) = 80A = 32-64i$

$10A = 4 - 8i$

$A = 2/5 - 4/5\, i$

$y_p = (2/5 - 4/5\, i)\, e^{x}(\cos(2x) + i\sin(2x))$

real part $= 2/5\, e^{x}\cos(2x) + 4/5\, e^{x}\sin(2x)$

General solution

$$y(x) = C_1e^{-x}\cos(2x) + C_2e^{-x}\sin(2x) + 2/5\, e^{x}\cos(2x) + 4/5\, e^{x}\sin(2x)$$

Suppose instead of $y'' + 2y' + 5y = 8e^{x}\cos(2x)$

we had $y'' + 2y' + 5y = 8e^{x}\sin(2x)$

take the same complex problem

$$y'' + 2y' + 5y = 8e^{x}e^{2ix} = 8e^{x}(\cos(2x) + i\sin(2x))$$

Now take the imaginary past.

$$y_p = (2/5 - 4/5\ i)\, e^{x}(\cos(2x) + i\sin(2x))$$

So solution is

$$y_p = 2/5\ e^{x}\sin(2x) - 4/5\ e^{x}\cos(2x)$$

General solution

$$y(x) = C_1e^{-x}\cos(2x) + C_2e^{-x}\sin(2x) + 2/5\ e^{x}\sin(2x) - 4/5\ e^{x}\cos(2x)$$

Back to original problem.

Find the general solution to $y'' + 2y' + 5y = 8e^{-x}\cos(2x)$

Homogeneous equation

$$y(x) = C_1e^{-x}\cos(2x) + C_2e^{-x}\sin(2x)$$

So $-1\pm 2i$ is all ready a solution

so $y_p = Axe^{(-1+2i)x}$

$$y_p' = A(1 + x(-1+2i))e^{(-1+2i)x}$$

$$y_p'' = A((-1+2i) + (-1+2i) + x(-1+2i)^2)e^{(-1+2i)}$$

$y_p'' + 2y_p' + 5y_p =$ The x terms will cancel.

$A(-2+4i\ +2\ )e^{(-1+2i)x} \qquad = 4iAe^{(-1+2i)x} = 8e^{(-1+2i)x}$

multiply by $-i$

$4A = -8i$

$A = -2i$

$y_p =$ real part $-2ixe^{-x}(\cos(2x) + i\sin(2x))$

$$y_p = 2xe^{-x}\sin(2x)$$

checks.

General solution

$$y(x) = C_1e^{-x}\cos(2x) + C_2e^{-x}\sin(2x) + 2xe^{-x}\sin(2x)$$

Suppose you had the same problem only the cos(2x) term was replaced by sin(2x). Then instead of taking the real part at the end you would take the imaginary part.

$$y_p = \text{imaginary part } -2ixe^{-x}(\cos(2x) + i\sin(2x))$$

$$y_p = -2xe^{-x}\cos(2x)$$

Review. Find a particular solution to an inhomogeneous linear partial differential equations with constant coefficients.

$Ly = P(D)y = h(x)$ $\quad$ $p(\lambda)$ = indicial polynomial

e.g. $Ly = y'' + 2y' + y$ $\quad$ $p(\lambda) = \lambda^2 + 2\lambda + 1$

$P(D)y = Ke^{\lambda x}$ $\quad$ if λ is not a root $y_p = Ae^{\lambda x}$ solve for A.

if λ is a root of order m then

$$y_p = Ax^me^{\lambda x}$$

$P(D)y = Ke^{ax}\cos(bx)$ or $Ke^{ax}\sin(bx)$

if $a\pm ib$ is not a root

$y_p = Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)$ $\quad$ solve for A and B or use complex method

$y_p = Ae^{(a+ib)x}$ $\quad$ solve for A and take real part for cos(bx) and imaginary part for sin(bx).

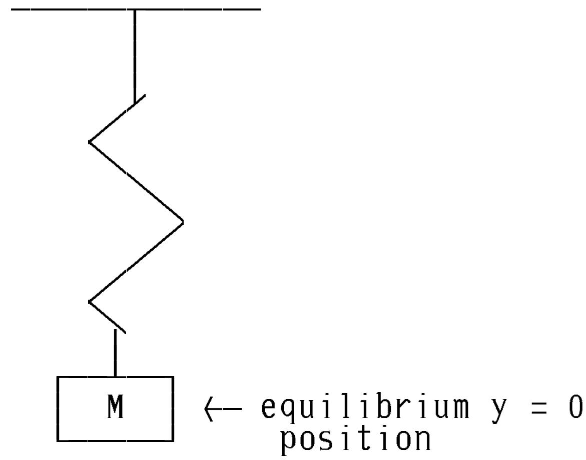
if $a\pm ib$ is a root then

$y_p = Axe^{ax}\cos(bx) + Bxe^{ax}\sin(bx)$ $\quad$ solve for A and B or use complex method

$y_p = Axe^{(a+ib)x}$ $\quad$ solve for A and take real part for cos(bx) and imaginary part for sin(bx).

Note in this case the (x) terms (e.g. $2x\ e^{2x}\cos(4x)$) will cancel.

Harmonic oscillator.
mass on a spring
undamped case.



Newton's law. $F = ma = m \frac{d^2y}{dt^2}$

Hooks law $F = -ky$

$$m \frac{d^2y}{dt^2} = -ky \qquad m \frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + (k/m)y = 0$$

indicial equation $\lambda^2 + (k/m) = 0$

$$\omega = \sqrt{k/m} \qquad \lambda^2 + \omega^2 = 0 \qquad \lambda = \pm i\omega$$

Solution is $y(t) = A \cos(\omega t) + B \sin(\omega t)$

$$y(t) = A \cos(\omega t + \varphi)$$

A is the amplitude equal maximum displacement

if $y(t) = A \cos(\omega t) + B \sin(\omega t)$ what is the amplitude

amplitude $= \sqrt{A^2 + B^2}$ What is max of $y(t)$?

$$y'(t) = -\omega A \sin(\omega t) + \omega B \cos(\omega t) = 0$$

$$A \sin(\omega t) = B \cos(\omega t)$$

$$A^2 \sin^2(\omega t) = B^2 \cos^2(\omega t) = B^2(1-\sin^2(\omega t))$$

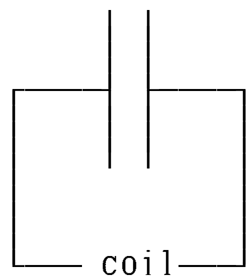
$$(A^2+B^2)\sin^2(\omega t) = B^2$$

$$\sin(\omega t) = \frac{\pm B}{\sqrt{A^2+B^2}} \qquad \cos(\omega t) = \frac{\pm A}{\sqrt{A^2+B^2}}$$

to get the max take the plus signs

$$\max = \frac{A^2+B^2}{\sqrt{A^2+B^2}} = \sqrt{A^2+B^2}$$

Electrical circuit



Damped Harmonic oscillator.

Viscous force $\vec{F} = -c\,\vec{v}$ $\vec{v}$ = velocity

So damped harmonic oscillator is

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt} \qquad m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

Indicial equation

$$m\lambda^2 + c\lambda + k = 0$$

$$\lambda = \frac{-c \pm \sqrt{c^2-4km}}{2m}$$

Underdamped. $c^2-4km < 0$ (typical as damping is small)

Let $\mu = \sqrt{4km-c^2}$

Solution is

$$y(t) = Ae^{-ct/2m}\cos(\mu t) + Be^{-ct/2m}\sin(\mu t)$$

$y(t) = Ae^{-ct/2m}\cos(\mu t+\varphi)$ draw picture.

Critically damped case $c^2 - 4km = 0$

double root $\lambda = -c/2m$

Solution is

$$y(t) = (A+Bt)e^{-ct/2m}$$

Take the case where $A > 0$ so $y(0) = A > 0$

If $B \geq 0$ system does not pass through equilibrium.

draw picture

Maximum occurs at when $y'(t) = 0$ at

$$y'(t) = -c(A+Bt)/2m \ e^{-ct/2m} + B\ e^{-ct/2m}$$

$y'(t) = 0$ when $2mB = cA + cBt$

$$t = (2mB-cA)/cB = 2m/c - A/B$$

if t negative then maximum at $t = 0$.

draw picture.

if $A > 0$ and $B < 0$ then system passes through equilibrium when $(A+Bt) = 0$ or $t = -A/B$