Inhomogeneous

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = h(x)$$

Find a particular solution y_p

General solution is $y_p + y_o$

where y is a solution to the homogeneous differential equation.

How to find a particular solution.

$$Ly = P(D)y = (D-\lambda_1)^{n_1}(D-\lambda_2)^{n_2}\cdots (D-\lambda_m)^{n_m}y = Ke^{\mu x}$$

if μ is not equal to any of the λ_i then

try $y_p = Ae^{\mu x}$ solve for A

$$A = \frac{K}{(\mu-\lambda_1)^{n_1}(\mu-\lambda_2)^{n_2}\cdots (\mu-\lambda_m)^{n_m}}$$

$P(\lambda)$ = indicial polynomial

$A = C/P(\mu)$

In summary to solve $P(D)y = Ke^{\mu x}$

and μ is not a root of the indicial equation

i.e. $P(\mu) \neq 0$

$$y_p = (C/P(\mu))e^{\mu x}$$

What if μ is a root of order k. Then

$$P(D)y = Ce^{\mu x}$$

$$y_p(x) = A\,x^ke^{\mu x}$$

Here you simply plug in this solution and solve for A.

Example $\quad y'' + 2y' + y = 6e^{-x}$

$$\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 \quad \lambda=-1 \text{ is a double root.}$$

try $\quad y = Ax^2e^{-x}$

$$y' = 2Axe^{-x} - Ax^2e^{-x}$$

$$y'' = 2Ae^{-x} - 2Axe^{-x} - 2Axe^{-x} + Ax^2e^{-x}$$

$y'' + 2y' + y =$ (x^2 terms cancel) (x terms cancel)

$$= 2Ae^{-x} = 6e^{-x} \quad A = 3$$

$$y_p(x) = 3x^2e^{-x}$$

General solution $\; y(x) = Ae^{-x} + Bxe^{-x} + 3x^2e^{-x}$

$Ly = P(D)y = C_1e^{ax}\cos(bx) + C_2a^{ax}\sin(bx)$

Assume $a\pm ib$ is not a root of $P(\lambda) = 0$

Try $Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)$

If $a\pm ib$ is already a root

Try $Axe^{ax}\cos(bx) + Bxe^{ax}\sin(bx)$

#11 Fall 2016

Find the general solution to $y'' + 2y' + 5y = 8e^{-x}\cos(2x)$

Too hard for now $\quad$ try $y'' + 2y' + 5y = 8e^{x}\cos(2x)$

Find the general solution to the homogeneous equation

$\lambda^2 + 2\lambda + 5 = 0 \qquad \lambda = \dfrac{-2 \pm \sqrt{4 - 4\cdot 5}}{2} = -1 \pm \sqrt{1-5} = -1 \pm 2i$

$C_1 e^{-x}\cos(2x) + C_2 e^{-x}\sin(2x)$

Try $Ae^{x}\cos(2x) + Be^{x}\sin(2x)$

plug in solve for A and B.

$y'' + 2y' + 5y = 8e^{x}e^{2ix} = 8e^{x}(\cos(2x) + i\sin(2x))$

take the real part

$y = Ae^{(1+2i)x}$

$y' = A(1+2i)e^{(1+2i)x}$

$y''(x) = A(1+2i)^2 e^{(1+2i)x} = A(-3+4i)e^{(1+2i)x}$

equation becomes

$A((-3+4i) + 2 + 4i + 5)e^{(1+2i)x} = 8e^{(1+2i)x}$

$A(4+8i) = 8$

$A(4+8i)(4-8i) = 32 - 64i$

$A(16+64) = 80A = 32-64i$

$10A = 4 - 8i$

$A = 2/5 - 4/5\, i$

$(2/5 - 4/5\, i)\, e^{x}(\cos(2x) + i\sin(2x))$

real part =

$2/5\, e^{x}\cos(2x) + 4/5\, e^{x}\sin(2x)$

General solution

$y(x) = C_1 e^{-x}\cos(2x) + C_2 e^{-x}\sin(2x) + 2/5\, e^{x}\cos(2x) + 4/5\, e^{x}\sin(2x)$

Back to original problem.

Find the general solution to $y'' + 2y' + 5y = 8e^{-x}\cos(2x)$

Homogeneous equation

$$y(x) = C_1e^{-x}\cos(2x) + C_2e^{-x}\sin(2x)$$

So $-1\pm 2i$ is all ready a solution

so $\quad y_p = Axe^{(-1+2i)x}$

$$y_p' = A(1 + x(-1+2i))e^{(-1+2i)x}$$

$$y_p'' = A((-1+2i) + (-1+2i) + x(-1+2i)^2)e^{(-1+2i)}$$

$y_p'' + 2y_p' + 5y_p =$ The x terms will cancel.

$$A(-2+4i +2)e^{(-1+2i)x} = 4iAe^{(-1+2i)x} = 8e^{(-1+2i)x}$$

multiply by $-i$

$$4A = -8i$$

$$A = -2i$$

$y_p =$ real part $-2ixe^{-x}(\cos(2x) + i\sin(2x))$

$$y_p = 2xe^{-x}\sin(2x)$$

checks.

General solution

$$y(x) = C_1e^{-x}\cos(2x) + C_2e^{-x}\sin(2x) + 2xe^{-x}\sin(2x)$$