Last time. Homogeneous linear n-th order differential equation with constant coefficients.

$$a_o\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdot + a_{n-1}\frac{dy}{dx} + a_ny = 0.$$

Indicial equation

$$a_o\lambda^n + a_1\lambda^{n-1} + \cdots\cdot + a_{n-1}\lambda + a_n = 0$$

If there are n distinct real roots $\lambda_1, \lambda_2, \cdots\cdot, \lambda_n$ so the differential equation can be written

$$a_o(D-\lambda_1)(D-\lambda_2)\cdots\cdot(D-\lambda_n)y = 0$$

Then the solution is $C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} + \cdots\cdot + C_ne^{\lambda_n x}$

boundary conditions

$$y(0) = K_0 \quad y'(0) = K_1 \quad y''(0) = K_2 \quad \cdots\cdot \quad y^{(n-1)}(0) = K_n$$

$$C_1 + C_2 + \cdots\cdot + C_n = K_0$$
$$\lambda_1C_1 + \lambda_2C_2 + \cdots\cdot + \lambda_nC_n = K_1$$
$$\lambda_1^2C_1 + \lambda_2^2C_2 + \cdots\cdot + \lambda_n^2C_n = K_2$$
$$\lambda_1^3C_1 + \lambda_2^3C_2 + \cdots\cdot + \lambda_n^3C_n = K_3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\lambda_1^nC_1 + \lambda_2^nC_2 + \cdots\cdot + \lambda_n^nC_n = K_n$$

Last time we did an example

$y'' + y' - 2y = 0 \quad y(0) = 2 \quad y'(0) = 3$

$(D+2)(D-1)y = 0$

$y = Ae^{x} + Be^{-2x}$

$y(0) = A + B$

$y'(0) = A-2B$

$y(0) = 2 \quad y'(0) = 3$

so $A + B = 2$

$A - 2B = 3$

$-A - B = -2$

$-3B = 1$ so $B = -1/3 \quad A = 7/3$

$$y(x) = (7/3)e^{x} - (1/3)e^{-2x}$$

Repeated real roots.

$D^2y = 0 \qquad y = A + Bx$

$(D-\lambda)^2y = 0 \qquad y = Ae^{\lambda x} + Bxe^{\lambda x}$

Example. $y'' + 4y' + 4y = 0 \quad y(0) = 2 \quad y'(0) = 3$

$(D+2)^2y = 0$

$y = Ae^{-2x} + Bxe^{-2x}$

$y(0) = A \quad y'(0) = 2A + B$

$A = 2 \qquad 2\cdot 2+B = 3 \quad B = -1$

$$y(x) = 2e^{-2x} - xe^{-2x}$$

Example. A double root and a single root.

$$(D-\lambda_1)(D-\lambda_2)^2y = 0$$

General solution $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} + C_3xe^{\lambda_2 x}$

Triple root.

$$(D-\lambda)^3y = 0 \quad y = Ae^{\lambda x} + Bxe^{\lambda x} + Cx^2e^{\lambda x}$$

$y(0) = A \quad y'(0) = \lambda A + B \quad y''(0) = \lambda^2A+2\lambda B+2C$

solve for A then B then C.

Complex roots.

Occur in complex conjugate pairs

$a \pm ib$

$C_1e^{ax}\cos(bx) + C_2e^{ax}\sin(bx)$

Example. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 2y = 0 \qquad y(0) = 1 \quad y'(0) = 0 \quad y''(0) = 0$

Indicial equation. $\lambda^3-\lambda^2 + 2 = 0$

(hint. $\lambda = -1$ is a root)

$$(\lambda+1)(\lambda^2-2\lambda + 2) = 0$$

$$\lambda = \frac{2 \pm \sqrt{2^2-4\cdot 2}}{2} = 1 \pm i$$

$y = Ae^{-x} + Be^{x}\cos(x) + Ce^{x}\sin(x) \qquad y(0) = A + B$

$y' = -Ae^{-x} + Be^{x}\cos(x) - Be^{x}\sin(x)$

$\qquad + Ce^{x}\cos(x) + Ce^{x}\sin(x) \qquad y'(0) = -A + B + C$

$y'' = Ae^{-x} + (B+C)e^{x}\cos(x) - (B+C)e^{x}\sin(x)$

$\qquad (-B+C)e^{x}\cos(x) + (-B+C)e^{x}\sin(x) \qquad y''(0) = A+2C$

$A + B = 1 \qquad 1$

$-A + B + C = 0 \qquad 2$

$A + 2C = 0 \qquad 3$

Multiply 2 by -1 $\quad A - B - C = 0$

$A + B = 1$

$2A - C = 1$

$2A + 4C = 0$

$-5C = 1 \qquad C = -1/5$

$A = 2/5 \qquad B = 3/5$

$B = 2/5$

Solution $y = (2/5)e^{-x} + (3/5)e^{x}\cos(x) - (1/5)e^{x}\sin(x)$

double pair

$$((D - a)^2 + b^2)^2$$

$$C_1e^{ax}\cos(bx) + C_2e^{ax}\sin(bx) + C_3xe^{ax}\cos(bx) + C_4xe^{ax}\sin(bx)$$

Inhomogeneous

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdot + a_{n-1}\frac{dy}{dx} + a_n y = h(x)$$

Find a particular solution y_p

General solution is $y_p + y$

where y is a solution to the homogeneous differential equation.

How to find a particular solution. Guess.

$$Ly = (D-\lambda_1)^{n_1}(D-\lambda_2)^{n_2}\cdots\cdot\ (D-\lambda_m)^{n_m}y = Ce^{\mu x}$$

if μ is not equal to any of the λ_i then

try $y_p = Ae^{\mu x}$

you get $(\mu-\lambda_1)^{n_1}(\mu-\lambda_2)^{n_2}\cdots\cdot\ (\mu-\lambda_m)^{n_m}\ A = C$

$$A = \frac{C}{(\mu-\lambda_1)^{n_1}(\mu-\lambda_2)^{n_2}\cdots\cdot\ (\mu-\lambda_m)^{n_m}}$$

$P(\lambda)$ = indicial polynomial

$A = C/P(\mu)$

In summary to solve $P(D)y = Ce^{\mu x}$

and μ is not a root of the indicial equation

i.e. $P(\mu) \neq 0$

$$y_p = (C/P(\mu))e^{\mu x}$$

What if μ is a root of order k. Then

$P(D)y = Ce^{\mu x}$

$$y_p(x) = A\,x^k e^{\mu x}$$

Here you simply plug in this solution and solve for A.