Differential equations

Linear n-th order.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\ a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x)$$

$a_k(x)$ are continuous functions of x

$a_0(x) \neq 0$

consider the equation $xy' + y = 0$

solution is $y = k/x$

you can not specify $y(0)$

$xy' - y = 0$

solution is $y = kx$

again you can not require $y(0) = 1$.

If $a_0(x) = 0$ complicated things happen which we will not discuss at this time.

usually we divide by $a_0(x)$ so $a_0(x) = 1$

If $F(x) = 0$ we call this the equation homogeneous $y = 0$ is a solution.

General theory. At a point x_0 you can freely specify

$y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$ and then the solution is uniquely determined.

Every n-th order differential equation is first order matrix equation. we assume $a_0(x) = 1$ (We can divide through by $a_0(x)$)

let $\vec{y} = (y(x), y'(x), \ldots, y^{(n-1)}(x))$

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-1} & -a_{n-3} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ \\ F \end{bmatrix}$$

↑ non homogeneous term

Importance you can prove there is a solution

$$\vec{y}(x) = \vec{y}(x_0) + \int_{x_0}^{x} A(t)\vec{y}(t) + \vec{c}(t)\, dt$$

Iterate

n-order linear differential equation with constant coefficients.

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = F$$

Homogeneous equation set $F = 0$

D = differential operator $\frac{d}{dx}$

$$(D^n + a_1 D^{n-1} + \cdots a_{n-1}D + a_n) = 0$$

$$(D-\lambda_1)^{k_1}(D-\lambda_2)^{k_2}\cdots(D-\lambda_m)^{\lambda_m} y = 0$$

Single real root $y = Ce^{\lambda_i x}$

Double real root $y = C_1 e^{\lambda_i x} + C_2 x e^{\lambda_i x}$

Triple real root $y = C_1 e^{\lambda_i x} + C_2 x e^{\lambda_i x} + C_3 x^2 e^{\lambda_i x}$

etc

Complex roots occur in pairs

$a \pm ib$ $C_1 e^{ax}\cos(bx) + C_2 e^{ax}\sin(bx)$

double pair

$$((D - a)^2 + b^2)^2$$

$$C_1 e^{ax}\cos(bx) + C_2 e^{ax}\sin(bx) + C_3 x e^{ax}\cos(bx) + C_4 x e^{ax}\sin(bx)$$

Discuss complex roots later.

Simplest case (single real roots)

differential equation factors

$$(D-\lambda_1)(D-\lambda_2)\cdots\cdots(D-\lambda_n)$$

Solution $C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots\cdots + C_n e^{\lambda_n x}$

boundary conditions

$$y(0) = K_0 \quad y'(0) = K_1 \quad y''(0) = K_2 \quad \cdots\cdots \quad y^{(n-1)}(0) = K_n$$

$$C_1 + C_2 + \cdots\cdots + C_n = K_0$$
$$\lambda_1 C_1 + \lambda_2 C_2 + \cdots\cdots + \lambda_n C_n = K_1$$
$$\lambda_1^2 C_1 + \lambda_2^2 C_2 + \cdots\cdots + \lambda_n^2 C_n = K_2$$
$$\lambda_1^3 C_1 + \lambda_2^3 C_2 + \cdots\cdots + \lambda_n^3 C_n = K_3$$

....

$$\lambda_1^n C_1 + \lambda_2^n C_2 + \cdots\cdots + \lambda_n^n C_n = K_n$$

Example

$$y'' + y' - 2y = 0 \qquad y(0) = 2 \quad y'(0) = 3$$

$$(D+2)(D-1)y = 0$$

$$y = Ae^{x} + Be^{-2x}$$

$$y(0) = A + B$$

$$y'(0) = A-2B$$

$$y(0) = 2 \quad y'(0) = 3$$

so $A + B = 2$

$$A - 2B = 3$$

$$-A - B = -2$$

$$-3B = 1 \quad \text{so } B = -1/3 \quad A = 7/3$$

$$y(x) = (7/3)e^{x} - (1/3)e^{-2x}$$