I have posted formulae for the exponential of (2x2) and (3x3)-matrices with real eigenvalues. Last time we discussed the case where A is a (3x3)-matrix with two eigenvalues λ_1 and λ_2 where λ_2 is a double root. (i.e. $p(\lambda) = -(\lambda-\lambda_2)(\lambda-\lambda_2)^2$.

If A is a real (3 X 3)-matrix with one real eigenvalue $(\lambda,\lambda,\lambda)$ so λ is a triple root of the characteristic equation

$$p(\lambda) = \det(A - \lambda I) = 0$$

then the exponential of A is given by

$$e^{tA} = e^{\lambda t}(I + t(A-\lambda I) + \tfrac{1}{2}t^2(A-\lambda I)^2).$$

#8 Spring 15

$$B = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \qquad \frac{d\vec{x}}{dt}(t) = B\,\vec{x}(t)$$

Find the general solution.

$$e^{tB} = ? \qquad \det(B - \lambda I) = (-1-\lambda)^3$$

$$\lambda = -1 \text{ triple root.}$$

$$e^{tB} = e^{-t}(I + t\,(B+I) + \tfrac{1}{2}t^2(B+I)^2)$$

$$B+I = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad (B+I)^2 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{tB} = e^{-t}\begin{bmatrix} 1 & 2t & 2t^2-t \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = e^{-t}\begin{bmatrix} a+(2b-c)t+2ct^2 \\ b + 2ct \\ c \end{bmatrix}$$

#4 S15

$$A = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

Find A^{11}

$$\det(A-\lambda I)) = (1-\lambda)^3$$

$$A = I + N \quad \text{so} \quad N = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{bmatrix}$$

$$N^2 = 0$$

$$A^{11} = (I+N)^{11} = I + 11N = \begin{bmatrix} -10 & 22 & -11 \\ -11 & 23 & -11 \\ -11 & 22 & -10 \end{bmatrix}$$

Today we start differential equation.

Linear n-th order.

$$a_o(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdot a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x)$$

$a_k(x)$ are continuous functions of x

$a_o(x) \neq 0$

consider the equation $xy' + y = 0$

solution is $y = k/x$

you can not specify $y(0)$

$xy' - y = 0$

solution is $y = kx$

again you can not require $y(0) = 1$.

If $a_o(x) = 0$ complicated things happen which we will not discuss at this time.

usually we divide by $a_o(x)$ so $a_o(x) = 1$

If $F(x) = 0$ we call this the equation homogeneous $y = 0$ is a solution.

General theory. At a point x_o you can freely specify

$y(x_o)$, $y'(x_o)$, $\cdots\cdot$, $y^{(n-1)}(x_o)$ and then the solution is uniquely determined.

Every n-th order differential equation is first order matrix equation. we assume $a_o(x) = 1$

let $\vec{y} = (y(x), y'(x), \cdots\cdot, y^{(n-1)}(x))$

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-1} & -a_{n-3} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ \\ F \end{bmatrix}$$

Importance you can prove there is a solution

$$\vec{y}(x) = \vec{y}(x_0) + \int_{x_0}^{x} A(t)\vec{y}(t) + \vec{c}(t)\,dt$$

Iterate

n-order linear differential equation with constant coefficients.

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = F$$

Homogeneous equation set $F = 0$

D = differential operator $\frac{d}{dx}$

$$(D^n + a_1 D^{n-1} + \cdots a_{n-1}D + a_n) = 0$$

$$(D-\lambda_1)^{k_1}(D-\lambda_2)^{k_2}\cdots(D-\lambda_m)^{\lambda_m}y = 0$$

Single real root $y = Ce^{\lambda_i x}$

Double real root $y = C_1 e^{\lambda_i x} + C_2 x e^{\lambda_i x}$

Triple real root $y = C_1 e^{\lambda_i x} + C_2 x e^{\lambda_i x} + C_3 x^2 e^{\lambda_i x}$

etc

Complex roots occur in pairs

$a \pm ib$ $\quad C_1 e^{ax}\cos(bx) + C_2 e^{ax}\sin(bx)$