A matrix A is nilpotent of order p if $A^p = 0$ and $A^{p-1} \neq 0$. A matrix A is nilpotent of order one if $A = 0$. Let

$$N_1 = [0],\ N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ N_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ etc.}$$

Note N_k^2 has 1's two up from the diagonal. For example,

$$N_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note N_k^3 has 1's three up from the diagonal. For example

$$N_4^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note $N_k^k = 0$.

Theorem if N is nilpotent of order p then

$$N = S \begin{bmatrix} N_{k_1} & 0 & 0 & \cdots & 0 \\ 0 & N_{k_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & N_{k_m} \end{bmatrix} S^{-1}$$

where $k_i \leq p$ for each $i = 1, \cdots, m$

and at least one $k_i = p$ (if not N is nilpotent of order p-1)

Now let

$$J_{\lambda 1} = [\lambda],\ J_{\lambda 2} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},\ J_{\lambda 3} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix},\ J_{\lambda 4} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \text{ etc.}$$

Note

$$J_{\lambda k} = \lambda I + N_k$$

$$\text{Now } e^{tN_4} = \begin{bmatrix} 1 & t & t^2/2! & t^3/3! \\ 0 & 1 & t & t^2/2! \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in general e^{tN_k}

has 1 on the diagonal

t one up from the diagonal

$t^2/2!$ two up from the diagonal

$t^3/3!$ three up from the diagonal

$t^k/k!$ k up from the diagonal

$$e^{tJ_{\lambda k}} = e^{t(\lambda I+N_k)} = e^{\lambda t}e^{tN_k}$$

So for example

$$e^{tJ_{\lambda 4}} = e^{t\lambda} \begin{bmatrix} 1 & t & t^2/2! & t^3/3! \\ 0 & 1 & t & t^2/2! \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The block $J_{\lambda k}$ can be repeated. Usually we group all the block J with the same λ together.

Jordan canonical form. Every (n x n)-matrix A is of the form

$$A = S \begin{bmatrix} J_{\lambda_1 k_1} & 0 & 0 & \cdots & 0 \\ 0 & J_{\lambda_2 k_2} & 0 & \cdots & 0 \\ & \cdots & \cdots\cdots\cdots & & \cdots \\ 0 & 0 & 0 & \cdots & J_{\lambda_m k_m} \end{bmatrix} S^{-1}$$

For each block $J_{\lambda k}$ λ is an eigenvalue of A and k = 1,2,····

The exponential is

$$A = S \begin{bmatrix} e^{tJ_{\lambda_1 k_1}} & 0 & 0 & \cdots & 0 \\ 0 \cdots\cdots & e^{tJ_{\lambda_2 k_2}} & 0 & \cdots & 0 \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & & \cdots \\ 0 \cdots\cdots & 0 & \cdots\cdots 0 & \cdots\cdots & e^{tJ_{\lambda_m k_m}} \end{bmatrix} S^{-1}$$

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$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -9 & 11 \end{bmatrix} \qquad \text{compute } e^{tA}$$

$$e^{tA} = \begin{bmatrix} e^{3t} & te^{3t} & t^2e^{3t}/2! & 0 & 0 \\ 0 & e^{3t} & te^{3t} & 0 & 0 \\ 0 & 0 & e^{3t} & 0 & 0 \\ 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & X & X \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & 6 \\ -9 & 11 \end{bmatrix} \quad \det(B-\lambda I) = \lambda^2 - tr(B)\lambda + \det(B) = \lambda^2 - 7\lambda + (-44+54)$$

$$\lambda^2 - 7\lambda + 10 = (\lambda-5)(\lambda-2)$$

If A is non defective with two eigenvalue λ_1 and λ_2 then

$$e^{tA} = (\lambda_2-\lambda_1)^{-1}(e^{\lambda_1 t}(A-\lambda_2 I) - e^{\lambda_2 t}(A-\lambda_1 I))$$

$$e^{tB} = (5-2)^{-1}(e^{5t}\begin{bmatrix} -6 & 6 \\ -9 & 9 \end{bmatrix} - e^{2t}\begin{bmatrix} -9 & 6 \\ -9 & 9 \end{bmatrix})$$

$$e^{tB} = (e^{5t}\begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix} - e^{2t}\begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix})$$

$$e^{tB} = \begin{bmatrix} 3e^{2t}-2e^{5t} & 2e^{5t}-2e^{2t} \\ 3e^{2t}-3e^{5t} & 3e^{5t}-2e^{2t} \end{bmatrix}$$

$$e^{0B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \frac{d}{dt} e^{tB} \Big|_{t=0} = \begin{bmatrix} 6-10 & 10-4 \\ 6-15 & 15-4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & 11 \end{bmatrix}$$

$$e^{tA} = \begin{bmatrix} e^{3t} & te^{3t} & t^2e^{3t}/2! & 0 & 0 \\ 0 & e^{3t} & te^{3t} & 0 & 0 \\ 0 & 0 & e^{3t} & 0 & 0 \\ 0 & 0 & 0 & 3e^{2t}-2e^{5t} & 2e^{5t}-2e^{2t} \\ 0 & 0 & 0 & 3e^{2t}-3e^{5t} & 3e^{5t}-2e^{2t} \end{bmatrix}$$

#5 Fall 14

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 0 & -5 \\ 4 & 0 & -4 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 3 & 0 & -5 \\ 4 & 0 & -4 \end{bmatrix}$$

e^{tA} = ? They asked for generalized eigenvectors.
multiple choice and they are not unique.

$$\det(A-\lambda I) = \det \begin{bmatrix} -\lambda & 0 & -1 \\ 3 & -\lambda & -5 \\ 4 & 0 & -4-\lambda \end{bmatrix} = -\lambda^2(\lambda+4) -4\lambda = -\lambda(\lambda^2+4\lambda+4) = -\lambda(\lambda+2)^2$$

roots are $\lambda = 0$ and $\lambda = -2$ double root.

is A defective

$$A+2I = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & -5 \\ 4 & 0 & -2 \end{bmatrix}$$

start row reduction.
Replace bottom row by itself minus twice top row

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

clear two equation so matrix is defective.

Use handy dandy formula

Exponential of a (2 x 2) matrix with repeated roots (λ,λ).

$$e^{tA} = e^{\lambda t}(I + t(A-\lambda I))$$

If A is a real (3×3)-matrix with real eigenvalues $(\lambda_1,\lambda_2,\lambda_2)$ so the characteristic equation

$$p(\lambda) = \det(A - \lambda I) = 0$$

has a single real root λ_1 and a double real root λ_2 then the exponential of A is given by

$$e^{tA} = e^{\lambda_2 t} I + te^{\lambda_2 t}(A-\lambda_2 I)$$
$$+ (\lambda_2-\lambda_1)^{-2}(e^{\lambda_1 t} - e^{\lambda_2 t} + (\lambda_2-\lambda_1)te^{\lambda_2 t})(A-\lambda_2 I)^2$$

Note if A is non defective then

$$e^{tA} = (\lambda_2-\lambda_1)^{-1}(e^{\lambda_2 t}(A-\lambda_1 I) - e^{\lambda_1 t}(A-\lambda_2 I))$$

$$\frac{d}{dt}e^{tA} = \lambda_2 e^{\lambda_2 t} I + \lambda_2 te^{\lambda_2 t}(A-\lambda_2 I) + e^{\lambda_2 t}(A-\lambda_2 I)$$
$$+ (\lambda_2-\lambda_1)^{-2}(\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t}+\lambda_2(\lambda_2-\lambda_1)te^{\lambda_2 t}+(\lambda_2-\lambda_1)e^{\lambda_2 t})(A-\lambda_2 I)^2$$

$$\frac{d}{dt}e^{tA} = \lambda_2 e^{\lambda_2 t} I +(1+\lambda_2 t)e^{\lambda_2 t}(A-\lambda_2 I)$$
$$+ (\lambda_2-\lambda_1)^{-2}(\lambda_1 e^{\lambda_1 t}- \lambda_1 e^{\lambda_2 t} +\lambda_2(\lambda_2-\lambda_1)te^{\lambda_2 t})(A-\lambda_2 I)^2$$

Note in powers of $(A-\lambda_2 I)$

$$AI = \lambda_2 I + (A-\lambda_2 I)$$

$$A(A-\lambda_2 I) = \lambda_2(A-\lambda_2 I) + (A-\lambda_2 I)^2$$

Now A satisfies its characteristic equation

$$(A-\lambda_1 I)(A-\lambda_2 I)^2 = 0$$

So $\quad A(A-\lambda_2 I)^2 = \lambda_1(A-\lambda_2 I)^2$

$$Ae^{tA} = \lambda_2 e^{\lambda_2 t} I + e^{\lambda_2 t}(A-\lambda_2 I) + t\lambda_2 e^{\lambda_2 t}(A-\lambda_2 I) + te^{\lambda_2 t}(A-\lambda_2 I)^2$$
$$+ (\lambda_2-\lambda_1)^{-2}\lambda_1(e^{\lambda_1 t} - e^{\lambda_2 t} + (\lambda_2-\lambda_1)te^{\lambda_2 t})(A-\lambda_2 I)^2$$

$$Ae^{tA} = \lambda_2 e^{\lambda_2 t} I + (1+t\lambda_2)e^{\lambda_2 t}(A-\lambda_2 I)$$
$$+ (\lambda_2-\lambda_1)^{-2}(\lambda_1 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} + \lambda_2(\lambda_2-\lambda_1)te^{\lambda_2 t})(A-\lambda_2 I)^2$$

$$\frac{d}{dt}e^{tA} = \lambda_2 e^{\lambda_2 t} I + (1+\lambda_2 t)e^{\lambda_2 t}(A-\lambda_2 I)$$
$$+ (\lambda_2-\lambda_1)^{-2}(\lambda_1 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} + \lambda_2(\lambda_2-\lambda_1)te^{\lambda_2 t})(A-\lambda_2 I)^2$$

So in our case $\lambda_1 = 0 \quad \lambda_2 = -2$

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 0 & -5 \\ 4 & 0 & -4 \end{bmatrix} \qquad A-\lambda_2 I = A + 2I = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & -5 \\ 4 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & -5 \\ 4 & 0 & -2 \end{bmatrix}$$

$$(A-\lambda_2 I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{tA} = e^{\lambda_2 t} I + te^{\lambda_2 t}(A-\lambda_2 I)$$
$$+ (\lambda_2-\lambda_1)^{-2}(e^{\lambda_1 t} - e^{\lambda_2 t} + (\lambda_2-\lambda_1)te^{\lambda_2 t})(A-\lambda_2 I)^2$$

$$e^{tA} = e^{-2t} I + te^{-2t}\begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & -5 \\ 4 & 0 & -2 \end{bmatrix} + \frac{1}{4}(1-e^{-2t} - 2te^{-2t})\begin{bmatrix} 0 & 0 & 0 \\ -8 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{tA} = e^{-2t} I + te^{-2t}\begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & -5 \\ 4 & 0 & -2 \end{bmatrix} + (1-e^{-2t} - 2te^{-2t})\begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & -3/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{tA} = \begin{bmatrix} e^{-2t}+2te^{-2t} & 0 & -te^{-2t} \\ -2+2e^{-2t}+7te^{-2t} & 1 & -3/4+3/4e^{-2t}-7/2te^{-2t} \\ 4te^{-2t} & 0 & e^{-2t}-2te^{-2t} \end{bmatrix}$$