If A is a non defective operator so it has a complete set of eigenvectors $A\vec{v_i} = \lambda_i \vec{v_i}$. A has eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ and for each eigenvalue λ_i there are p_i linearly independent eigenvectors. $\vec{v_{ij}}$ for $j = 1, \cdots, p_i$. Since the eigenvectors span $\mathbb{R}^n$ we have $p_1 + p_2 + \cdots + p_m = n$.

There are spectral projections E_i so that $E_i^2 = E_i$ and $E_i E_j = \delta_{ij} E_i$ (δ_{ij} is the Kronecker delta so $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$). We have

$$AE_i = \lambda_i E_i \qquad E_1 + E_2 + \cdots + E_m = I$$

$$e^{tA} = e^{\lambda_1 t} E_1 + e^{\lambda_2 t} E_2 + \cdots + e^{\lambda_m t} E_m$$

$$E_i = (\lambda_i - \lambda_1)^{-1}(A - \lambda_1 I)(\lambda_i - \lambda_2)^{-1}(A - \lambda_2 I) \cdots (\lambda_i - \lambda_m)^{-1}(A - \lambda_m I)$$

with the $(\lambda_i - \lambda_i)^{-1}(A - \lambda_i I)$ term ommited.

If $AA^T = A^T A$ then A in said to be normal. Such A are non defective. Note if A is symmetric $a_{ij} = a_{ji}$ the A is normal and if A is real then A has real spectrum. If A is antisymmetric so $a_{ij} = - a_{ji}$ and A is real then A has pure imaginary spectrum. Every real matrix can be written as the sum A = B + C where B is symmetric and C is antisymmetric. The matrix A is normal if and only if B and C commute.

If A is an (n x n)-matrix then A satisfies its characteristic equation.

$$p(\lambda) = \det(A - \lambda I) \qquad \text{then } p(A) = 0$$

i.e.

$$(-A)^n + tr(A)(-A)^{n-1} + c_2(-A)^{n-2} + \cdots + c_{n-1}(-A) + \det(A)I = 0$$

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{p_1}(\lambda - \lambda_2)^{p_2} \cdots (\lambda - \lambda_m)^{p_m}$$

note $p_1 + p_2 + \cdots + p_m = n$ (A is an (n x n)-matrix)

So $(A - \lambda_1 I)^{p_1}(A - \lambda_2 I)^{p_2} \cdots (A - \lambda_m I)^{p_m} = 0$

An matrix A is non defective if and only if

$$(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_m I) = 0.$$

so the minimal polynomial $p_m(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_m)$

Notice if A is non defective and has two eigenvalues λ_1 and λ_2 then

$$e^{tA} = (\lambda_2-\lambda_1)^{-1}(e^{\lambda_1 t}(A-\lambda_2 t) - e^{\lambda_2 t}(A-\lambda_1 I))$$

The minimal polynomial is the polynomial of lowest degree so that $p_m(A) = 0$.

Defective matrix.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

characteristic polynomial

$$p(\lambda) = \det(A-\lambda I) = \lambda^2 - tr(A)\lambda + \det(A) = \lambda^2$$

$\lambda = 0$ eigenvalue.

$$A-0I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A\vec{v} = 0 \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad y = 0$$

The space of eigenvectors for $\lambda = 0$ is one dimensional

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note $A^2 = 0$. A matrix N so that $N^p = 0$ is called nilpotent. The order p is the first power so that $N^p = 0$ so $N^k \neq 0$ for $k < p$.

What are the nilpotent (2 x 2)-matrices. $N^2 = 0$

A (2 x 2) matrix can not be nilpotent or order 3.

Another example $N^2 = 0$

If $N^2 = 0$

$tr(N) = 0 \quad det(N) = 0$

$$N = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \qquad det(N) = - a^2 - bc$$

$$bc = -a^2$$

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & -a \\ 1/a & -1 \end{bmatrix} \begin{bmatrix} 1 & -a \\ 1/a & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Every (2 x 2)-matrix N so that $N \neq 0$ and $N^2 = 0$

is of the form $S\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S^{-1}$

$\vec{v_1} = N\vec{v_2}$ with $\vec{v_2}$ choosen so that $\vec{v_1} \neq 0$

then $N\vec{v_1} = 0$

So in terms of $\vec{v_1}$ and $\vec{v_2}$ we have

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$N = S \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S^{-1} \qquad S = \begin{bmatrix} (\vec{v_1})_1 & (\vec{v_2})_1 \\ (\vec{v_1})_2 & (\vec{v_2})_2 \end{bmatrix}$$

$$e^{tN} = I + tN$$

$$(I+tN)(I+sN) = (I+(t+s)N)$$

If A is a (2 x 2)-matrix with a repeated eigenvalue λ

$A = \lambda I + N$ Note $e^{t(A+B)} = e^{tA}e^{tB}$ if A and B commute meaning $AB = BA$. Since $IA = AI$ we have

$$e^{tA} = e^{t(\lambda I+N)} = e^{t\lambda I}e^{tN} = e^{\lambda t}(I + tN + \frac{t^2}{2!} N^2 + \cdots) = e^{\lambda t}(I+tN)$$

Formula for e^{tA} where A is a (2 x 2)-matrix with repeated eigenvalue λ. Note $(A-\lambda I)^2 = 0$

$$e^{tA} = e^{\lambda t}(I + t(A-\lambda I)) \qquad \text{Note } e^{tA} = e^{t\lambda}e^{t(A-\lambda I)}$$

What about (3 x 3)-matrices. With three repeated zero eigenvalues.

$$\det(A-\lambda I) = -\lambda^3 + \operatorname{tr}(A)\lambda^2 - c\lambda + \det(A) = 0$$

$$\operatorname{tr}(A) = 0 \quad \det(A) = 0 \text{ and } c = 0$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$$c = a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{12}a_{21}-a_{13}a_{31}-a_{23}a_{32}$$

example

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example of a matrix that is nilpotent of order 4

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Every nilpotent matrix is of the form $N = SJS^{-1}$

J is a direct sum of N_i above

e.g

$$J = \begin{bmatrix} N_1 & 0 & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 & 0 \\ 0 & 0 & N_3 & 0 & 0 \\ 0 & 0 & 0 & N_4 & 0 \\ 0 & 0 & 0 & 0 & N_5 \end{bmatrix}$$

N_i is of the form given below

order 1 2 3 4

$$N_i = [0] \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{etc.}$$

Note if N is nilpotent of order p then

$$e^{tN} = I + tN + \frac{t^2}{2!}N^2 + \cdots + \frac{t^{p-1}}{(p-1)!} N^{p-1}$$

Notice $e^{t(\lambda I+N)} = e^{t\lambda}e^{tN}$

$$\text{e.g. if} \quad N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e^{t(\lambda I+N)} = e^{t} \begin{bmatrix} 1 & t & t^2/2! & t^3/3! \\ 0 & 1 & t & t^2/2! \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$