Last test. Median was 73. Roughly a B.

Suppose A is real (n x n)-matrix and $\mathfrak{N}$ is an m-dimensional subspace of $\mathbb{R}^n$. Then there are m linearly independent vectors $\{\vec{v}_1, \cdots, \vec{v}_m\}$ so that every vector $\vec{w} \in \mathfrak{N}$ is a linear combination of the $\vec{v}$'s so

$$\vec{w} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_m\vec{v}_m.$$

We say $\mathfrak{N}$ is an invariant subspace for A if A maps $\mathfrak{N}$ into itself so if $\vec{w} \in \mathfrak{N}$ then $A\vec{w} \in \mathfrak{N}$. We can denote this another way by writing $A\mathfrak{N} \subset \mathfrak{N}$. Now if $\mathfrak{N}$ is an invariant subspace for A then we can consider the restriction of A to $\mathfrak{N}$ which we denote by $A|\mathfrak{N}$. Now the restriction of A to $\mathfrak{N}$ or $A|\mathfrak{N}$ can be represented by an (m x m)-matrix so

$$A\vec{v}_j = a_{1j}\vec{v}_1 + a_{2j}\vec{v}_2 + \cdots + a_{mj}\vec{v}_m$$

Now if we want to compute

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots.$$

we see that since each term maps $\mathfrak{N}$ into itself so $e^{tA}$ maps $\mathfrak{N}$ into itself. Now the restriction of $e^{tA}$ to $\mathfrak{N}$ can be computed from the matrix $A|\mathfrak{N}$. So if we want to solve the problem

$$\frac{d\vec{x}}{dt}(t) = A\vec{x}(t) \qquad \text{and} \qquad \vec{x}(0) \in \mathfrak{N}$$

then we only have to compute the exponential of $A|\mathfrak{N}$ which is often much easier than computing the exponential of A on the whole space.

Consider #5 Fall 2013

Let $\vec{x}(t)$ be a 4-vector function satisfying $\frac{d\vec{x}}{dt}(t) = \begin{bmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \vec{x}(t)$
and

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Find } \vec{x}(\pi/8)$$

Compute $A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ $\quad A \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ So in $\mathfrak{N} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$

$A\mathfrak{N} \subset \mathfrak{N}$ and $A|\mathfrak{N}$ is given by the (2 x 2)-matrix

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$e^{t\begin{bmatrix} a & b \\ -b & a \end{bmatrix}} = e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$$

So we only need to consider the easier problem

$$e^{tA} \quad \text{with } A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\text{so} \quad e^{tA} = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}$$

$$\begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x}(\pi/8) = \begin{bmatrix} \sqrt{1/2} \\ -\sqrt{1/2} \\ 0 \\ 0 \end{bmatrix}$$

Now suppose $\mathfrak{N}_1, \mathfrak{N}_2$ are subspaces of $\mathbb{R}^n$ which span $\mathbb{R}^n$ and the intersection of $\mathfrak{N}_1$ with $\mathfrak{N}_2$ contains only the vector $\vec{0}$. Then we say $\mathbb{R}^n$ is the direct sum of $\mathfrak{N}_1$ and $\mathfrak{N}_2$. Now suppose A maps $\mathfrak{N}_1$ into itself and A maps $\mathfrak{N}_2$ into itself so

$$A\mathfrak{N}_1 \subset \mathfrak{N}_1 \qquad \text{and} \qquad A\mathfrak{N}_2 \subset \mathfrak{N}_2$$

Then we can form a basis $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_r}$ for $\mathfrak{N}_1$ and $\vec{v}_{r+1}, \cdots, \vec{v_n}$ for $\mathfrak{N}_2$ and the basis $\vec{v_1}, \cdots\cdot, \vec{v_n}$ is a basis for $\mathbb{R}^n$. When we express A as a (n x n)-matrix we see A is of the form

$$\begin{bmatrix}
a_{11} & a_{12} \cdots & a_{1r} & 0 & 0 & & 0 \\
a_{21} & a_{22} \cdots & a_{2r} & 0 & 0 & & 0 \\
\cdots & \cdots & \cdots & & & & \\
a_{r1} & a_{r2} \cdots & a_{rr} & 0 & 0 & & 0 \\
0 & 0 & 0 & b_{11} & b_{12} & \cdots & b_{1p} \\
0 & 0 & 0 & b_{11} & b_{12} & \cdots & b_{1p} \\
 & & & & & & \\
0 & 0 & 0 & b_{11} & b_{12} & \cdots & b_{1p}
\end{bmatrix}$$

technically $b_{ij} = a_{(r+i)(r+j)}$

We call this the direct sum of A and B $\quad A \oplus B$

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

Note

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} AC & 0 \\ 0 & BD \end{bmatrix}$$

$$e^t \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} e^{tA} & 0 \\ 0 & e^{tB} \end{bmatrix}$$

$$A \oplus B \oplus C$$

$$e^{t(A \oplus B \oplus C)} = e^{tA} \oplus e^{tB} \oplus e^{tC}$$

So for example

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} e^{t}\cos(2t) & -e^{t}\sin(2t) & 0 \\ e^{t}\sin(2t) & e^{t}\cos(2t) & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

$$A = \begin{bmatrix} X & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & \multicolumn{2}{c}{e^{tB}} \\ 0 & & \end{bmatrix} \qquad e^{tB} = 2 \times 2 \text{ matrix}$$

Also

$$A = \begin{bmatrix} X & 0 & X \\ 0 & Y & 0 \\ X & 0 & X \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} Y & 0 & Y \\ 0 & e^{\lambda t} & 0 \\ Y & 0 & Y \end{bmatrix}$$

$$\begin{bmatrix} y_{11} & y_{13} \\ y_{31} & y_{33} \end{bmatrix} = \exp(t \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

So if $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} e^{t}\cos(2t) & 0 & -e^{t}\sin(2t) \\ 0 & e^{2t} & 0 \\ e^{t}\sin(2t) & 0 & e^{t}\cos(2t) \end{bmatrix}$

The (4 x 4) case

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} e^{tB} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & e^{tC} & \\ 0 & 0 & & \end{bmatrix}$$

$$\exp\left(t\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{bmatrix}\right) = \begin{bmatrix} e^{tA} & 0 & 0 & 0 \\ 0 & e^{tB} & 0 & 0 \\ 0 & 0 & e^{tC} & 0 \\ 0 & 0 & 0 & e^{tD} \end{bmatrix}$$

If A is a non defective operator so it has a complete set of eigenvectors $A\vec{v}_i = \lambda_i\vec{v}_i$. A has eigenvalues $\{\lambda_1,\lambda_2,\cdots\cdot,\lambda_m\}$ and for each eigenvalue $\lambda_i$ there are $p_i$ linearly independent eigenvectors. $\vec{v}_{ij}$ for $j = 1,\cdots\cdot,p_i$. Since the eigenvectors span $\mathbb{R}^n$ we have $p_1+p_2+\cdots\cdot+p_m = n$.

There are spectral projections $E_i$ so that $E_i^2 = E_i$ and $E_iE_j = \delta_{ij}E_i$ ($\delta_{ij}$ is the Kronecker delta so $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$). We have

$$AE_i = \lambda_i E_i \qquad E_1 + E_2 + \cdots\cdot + E_m = I$$

$$e^{tA} = e^{\lambda_1 t}E_1 + e^{\lambda_2 t}E_2 + \cdots\cdot + e^{\lambda_m t}E_m$$

$$E_i = (\lambda_i-\lambda_1)^{-1}(A-\lambda_1 I)(\lambda_i-\lambda_2)^{-1}(A-\lambda_2 I)\cdots\cdot(\lambda_i-\lambda_m)^{-1}(A-\lambda_m I)$$

with the $(\lambda_i-\lambda_i)^{-1}(A-\lambda_i I)$ term ommited.