Last time

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)(-\lambda) + 2 = 0 \qquad \lambda^2 - tr(A)\lambda + \det(A) = 0.$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} \qquad 1 \pm i$$

Find eigenvector $\lambda = 1 + i$

$$A - \lambda I = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}$$

the top equation is the bottom equation time 1-i

$$(-1-i)(1-i) = -1 \; -1 \; -i + i = -2$$

$ax+by = 0$ solution is $x=-b$ $y=a$

$x= 1+i$ $y = 1$

solution is $e^{(1+i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

$$e^{(1+i)t} = e^t e^{it} = e^t(\cos(t)+i\sin(t))$$

$$(1+i)e^t(\cos(t)+i\sin(t)) = e^t(\cos(t)-\sin(t) + i(\sin(t)+\cos(t)))$$

The solution is

$$e^{(1+i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} e^t(\cos(t)-\sin(t)) + ie^t(\sin(t)+\cos(t)) \\ e^t\cos(t) \quad + \quad ie^t\sin(t) \end{bmatrix}$$

Real and imaginay part are both solutions

$$e^t \begin{bmatrix} \cos(t)-\sin(t) \\ \cos(t) \end{bmatrix} \qquad e^t \begin{bmatrix} \sin(t)+\cos(t) \\ \sin(t) \end{bmatrix}$$

$\vec{v_1}(t) \quad \vec{v_2}(t)$

set $t = 0$ you have $\vec{v_1}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v_2}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$S(t) = e^t \begin{bmatrix} \cos(t)-\sin(t) & \sin(t)+\cos(t) \\ \cos(t) & \sin(t) \end{bmatrix} \quad S(0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

first column is $\vec{v_1}(t)$ second column is $\vec{v_2}(t)$

$$e^{tA} = S(t)S(0)^{-1}$$

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \det(A) = -1$$

$$S^{-1} = (-1) \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$e^{tA} = e^t \begin{bmatrix} \cos(t)-\sin(t) & \sin(t)+\cos(t) \\ \cos(t) & \sin(t) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= e^t \begin{bmatrix} \cos(t)+\sin(t) & -2\sin(t) \\ \sin(t) & \cos(t)-\sin(t) \end{bmatrix}$$

check set $t = 0$ you get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

take the derivative at $t = 0$

$$\begin{bmatrix} 1+1 & -2 \\ 1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$$

Problem in complex numbers.

$(3-4i)x = 5+10i$ solve for x.

Multiply both sides time $3+4i$ complex conjugate of $3-4i$

$$(3^2+4^2)x = (3+4i)(5+10i) = 5(3+4i)(1+2i) = 5((3\text{x}1-4\text{x}2) + i(3\text{x}2+4\text{x}1))$$

$25x = 5(-5+10i)$

$x = -1+2i$ check $(3-4i)(-1+2i) = -3+8 + i(4+6) = -5+10i$

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$$A = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{find } e^{tA}$$

Note if the 2 was not there

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} \cos(t) & 0 & -\sin(t) \\ 0 & e^{t} & 0 \\ \sin(t) & 0 & \cos(t) \end{bmatrix}$$

But the 2 is there. So we must deal with it.

1. find eigenvalues

$$\det(A - \lambda I) = \begin{bmatrix} -\lambda & 2 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = \lambda^2(1-\lambda) + (1-\lambda)$$

$$(1-\lambda)\,(\lambda^2+1)$$

eigenvalue $1 \; i \; -i$

Find the eigenvectors

$$(A-I)\vec{v} = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

bottom equation

$x - z = 0 \quad x = 1 \; z = 1$

top equation is $-1 + 2y - 1 = 0$ so $y = 1$

$S(t) = [e^{\lambda_1 t}v_1, \text{ real}, \text{ imaginary}]$ real and imaginary parts of the complex eigenvector.

$$S(t) = \begin{bmatrix} e^t & & \\ e^t & & \\ e^t & & \end{bmatrix}$$

take the eigenvalue i

$$A - iI = \begin{bmatrix} -i & 2 & -1 \\ 0 & 1-i & 0 \\ 1 & 0 & -i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$(1-i)y = 0$ middle equation so $y = 0$

top equation $-ix - z = 0$ $x=1$ $z=-i$

$$e^{it}\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = (\cos(t) + i\sin(t))\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = \begin{bmatrix} \cos(t) + i\sin(t) \\ \\ \sin(t) - i\cos(t) \end{bmatrix}$$

real part $\begin{bmatrix} \cos(t) \\ 0 \\ \sin(t) \end{bmatrix}$ imaginary part $\begin{bmatrix} \sin(t) \\ 0 \\ -\cos(t) \end{bmatrix}$

$$S(t) = \begin{bmatrix} e^t & \cos(t) & \sin(t) \\ e^t & 0 & 0 \\ e^t & \sin(t) & -\cos(t) \end{bmatrix}$$

$$S = S(0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \qquad \det(S) = 1$$

$$S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad \det(S) = 1$$

$$e^{tA} = \begin{bmatrix} e^t & \cos(t) & \sin(t) \\ e^t & 0 & 0 \\ e^t & \sin(t) & -\cos(t) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \cos(t) & e^t-\cos(t)+\sin(t) & -\sin(t) \\ 0 & e^t & 0 \\ \sin(t) & e^t-\sin(t)-\cos(t) & \cos(t) \end{bmatrix}$$

Maple checks