Last time. How to compute e^{tA}

Find the eigenvalues which are solution to the equation

$$p(\lambda) = \det(A-\lambda I) = 0.$$

Given the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$

Find the eigenvectors

$$A\vec{v_1} = \lambda_1 \vec{v_1} \quad A\vec{v_2} = \lambda_2 \vec{v_2} \quad \cdots \quad A\vec{v_n} = \lambda_n \vec{v_n}$$

Possible some eigenvalues have two or more eigenvectors as we saw in the previous lecture. Form

$$S(t) = \begin{bmatrix} e^{t\lambda_1}(\vec{v_1})_1 & e^{t\lambda_2}(\vec{v_2})_1 & \cdots & e^{t\lambda_n}(\vec{v_n})_1 \\ e^{t\lambda_1}(\vec{v_1})_2 & e^{t\lambda_2}(\vec{v_2})_2 & \cdots & e^{t\lambda_n}(\vec{v_n})_2 \\ \cdots & \cdots & \cdots & \cdots \\ e^{t\lambda_1}(\vec{v_1})_n & e^{t\lambda_2}(\vec{v_2})_n & \cdots & e^{t\lambda_n}(\vec{v_n})_n \end{bmatrix}$$

$$e^{tA} = S(t)S(0)^{-1}$$

(2 x 2)-matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

$$p(\lambda) = \det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$

$$= \lambda^2 - tr(A)\lambda + \det(A)$$

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

If the characteristic polynomial has two real roots λ_1 and λ_2. Note $\lambda_1 + \lambda_2 = tr(A)$ and $\lambda_1\lambda_2 = det(A)$.

$$e^{tA} = e^{\lambda_1 t}(\lambda_1-\lambda_2)^{-1}(A-\lambda_2 I) + e^{\lambda_2 t}(\lambda_2-\lambda_1)^{-1}(A-\lambda_1 I)$$

$$= (\lambda_1-\lambda_2)^{-1}((e^{\lambda_1 t}-e^{\lambda_2 t})A + (\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t})I$$

Now the spectrum of a real matrix can be complex.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad det(A - \lambda I) = \lambda^2 + 1$$

$$A^2 = - I \qquad A^3 = -A \quad A^4 = I \quad A^5 = A$$

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\cdot$$

$$e^{tA} = I + tA - \frac{t^2}{2!}I - \frac{t^3}{3!}A + \frac{t^4}{4!}I + \frac{t^5}{5!}A - \cdots\cdot$$

$$= \cos(t)\, I + \sin(t)\, A = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

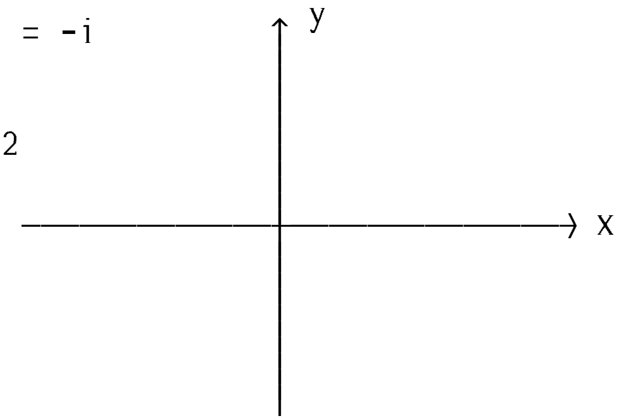
$$\lambda^2 + 1 = 0 \text{ then } \lambda = \pm i$$

We need complex numbers.

Complex numbers

$$x^2 = -1 \quad \text{so} \quad x = i \qquad x = -i$$

$$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$$



$$z_1 + z_2 = (x_1 + y_1) + i(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = z_2 z_1$$

$$\sqrt{i} = \pm \frac{1}{\sqrt{2}}(1 + i)$$

Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} +$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} +$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} +$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} -$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}$$

$$\cos(\theta+\varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$

$$\sin(\theta+\varphi) = \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\frac{d}{d\theta} e^{i\theta} = i\ e^{i\theta} \quad \text{gives} \quad \frac{d}{d\theta}\cos(\theta) = -\sin(\theta)$$

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$$

for example $\cos^2(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

absolute value $|z| = \sqrt{x^2 + y^2}$

complex conjugate

$$\bar{z} = z^* = x - iy$$

$$\bar{z}z = x^2 + y^2 \qquad |z| = \sqrt{\bar{z}z}$$

$$z = x + iy \quad = |z|\ e^{i\theta}$$

multiplication

$$z_1 = x_1 + iy_1 \quad = |z_1|\ e^{i\theta_1}$$

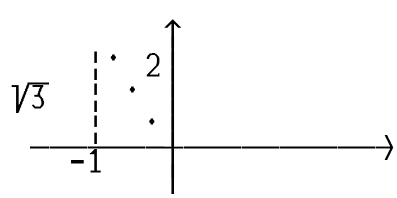
$$z_2 = x_2 + iy_2 \quad = |z_2|\ e^{i\theta_2}$$

$$z_1 z_2 = |z_1||z_2|\ e^{i(\theta_1+\theta_2)} \qquad e^{x+iy} \quad = e^x\cos(y) + i\ e^x\sin(y)$$

Problem. Find the cube roots of one

$$z^3 = 1 \qquad re^{i\theta} \qquad 3\theta = 0, 2\pi$$

$$e^{i2\pi/3} \quad = \cos(2\pi/3) + i\ \sin(2\pi/3)$$



$= -\frac{1}{2} + \frac{1}{2}\sqrt{3}\ i$ note sum of roots = 0

also $e^{-i2\pi/3} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}\ i$

Spectrum Suppose V a real linear space. We can make V a vector space over the complex numbers by considering vectors of the form $\vec{x} + i\vec{y}$ Then $(a+ib)(\vec{x} + i\vec{y}) = a\vec{x}-b\vec{y} + i(b\vec{x}+a\vec{y})$

Consider $\mathbb{R}^n = (x_1,x_2,\cdots,x_n)$ Complexification

$\mathbb{C}^n = (z_1,z_2,\cdots,z_n)$

Example.

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$$

$\det(A-\lambda I) = (2-\lambda)(-\lambda) + 2 = 0 \quad \lambda^2-\mathrm{tr}(A)\lambda + \det(A) = 0.$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} \qquad 1 \pm i$$

Find eigenvector $\lambda = 1 + i$

$$A - \lambda I = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}$$

the top equation is the bottom equation time 1-i

$(-1-i)(1-i) = -1 -1 -i + i = -2$

$ax+by = 0$ solution is $x=-b$ $y=a$

$x= 1+i$ $y = 1$

solution is $e^{(1+i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

$$e^{(1+i)t} = e^t e^{it} = e^t(\cos(t)+i\sin(t))$$

$$(1+i)e^t(\cos(t)+i\sin(t)) = e^t(\cos(t)-\sin(t) + i(\sin(t)+\cos(t)))$$

The solution is

$$e^{(1+i)t}\begin{bmatrix}1+i\\ 1\end{bmatrix} = \begin{bmatrix} e^t(\cos(t)-\sin(t)) + ie^t(\sin(t)+\cos(t)) \\ e^t\cos(t) \quad + \quad ie^t\sin(t)\end{bmatrix}$$

Real and imaginay part are both solutions

$$e^t\begin{bmatrix}\cos(t)-\sin(t)\\ \cos(t)\end{bmatrix} \qquad e^t\begin{bmatrix}\sin(t)+\cos(t)\\ \sin(t)\end{bmatrix}$$

$$\vec{v_1}(t) \qquad \vec{v_2}(t)$$

set $t = 0$ you have $\vec{v_1}(0) = \begin{bmatrix}1\\1\end{bmatrix}$ $\quad \vec{v_2}(0) = \begin{bmatrix}1\\0\end{bmatrix}$

$$S(t) = e^t\begin{bmatrix}\cos(t)-\sin(t) & \sin(t)+\cos(t)\\ \cos(t) & \sin(t)\end{bmatrix} \qquad S(0) = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$$

first column is $\vec{v_1}(t)$ second column is $\vec{v_2}(t)$

$$e^{tA} = S(t)S(0)^{-1}$$

$$S = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \qquad \det(A) = -1$$

$$S^{-1} = (-1)\begin{bmatrix}0 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$$

$$e^{tA} = e^t\begin{bmatrix}\cos(t)-\sin(t) & \sin(t)+\cos(t)\\ \cos(t) & \sin(t)\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$$

$$= e^t\begin{bmatrix}\cos(t)+\sin(t) & -2\sin(t)\\ \sin(t) & \cos(t)-\sin(t)\end{bmatrix}$$

check set $t = 0$ you get $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

take the derivative at $t = 0$

$$\begin{bmatrix}1+1 & -2\\ 1 & 1-1\end{bmatrix} = \begin{bmatrix}2 & -2\\ 1 & 0\end{bmatrix}$$