Rank Nullity of a matrix.

A = (m x n)-matrix. A $\mathbb{R}^n \longrightarrow \mathbb{R}^m$

Null space of A set of $\vec{x}$ so that $A\vec{x} = 0$.

Null space is a subspace.

Range of A the set of vectors in $\mathbb{R}^m$ of the form $A\vec{x}$.

Subspace. Nullity of A = dimension of the null space of A

Rank of A is the dimension of the range of A.

Note the columns of A are vectors in the range of A so the rank of A is the dimension of the column space of A.

To find a basis for the column space of A take the transpose of A^T

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(m x n) matrix

$$A^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

(n x m) matrix

And row reduce.

Theorem $\text{rank}(A) = \text{rank}(A^T)$.

Not in the course.

A is rank one if $a_{ij} = b_i c_j$

Note if A is rank one if and only if A^T is rank one.

Every matrix is the sum of rank one matrices.
The rank of a matrix is the least number of terms need to write A as sum of rank one matrices.
A matrix is of rank n if and only if

$$a_{ij} = \sum_{k=1}^{n} (\vec{b_k})_i (\vec{c_k})_j \qquad \vec{b_k} \in \mathbb{R}^m \quad \vec{c_k} \in \mathbb{R}^n$$

where the vectors $\vec{b_k}$ are linearly independent
the vectors $\vec{c_k}$ are linearly independent

Now it is clear $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.
Important. The transpose reverses the order of multiplication.

$$(AB)^T = B^T A^T$$

To compute the rank of a matrix row reduce and count the non zero rows.
Theorem. If A is an $(m \times n)$-matrix then $\mathrm{rank}(A) + \mathrm{null}(A) = n$.
Question. Suppose A and B are $(n \times n)$-matrices of rank p and q, respectively. What is the rank of AB.

Answer. $\mathrm{rank}(AB) \leq \min(p,q)$.

If A is rank p then $\mathrm{rank}(AC) \leq p$
because the range of AC is contained in the range of A
If A is rank p then $\mathrm{rank}(CA) \leq p$ because $(CA)^T = A^T C^T$
and the range of $A^T C^T$ is contained in the range of A^T.
How small can the rank of AB be?

$\mathrm{rank}(AB) \geq p + q - n$

null space of A has dimension n-p

range of B has dimension q.

So put as many vectors in the range of B in the null space of A.
vectors left over is q-(n-p) So the range of AB must be at least p+q-n.

Consider the space of all polynomials of degree three or less.

This is a four dimensional space.

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_o$$

basis $1, x, x^2, x^3$.

$$(Tp)(x) = p'(x) \qquad (Tp)(x) = 3a_3x^2 + 2a_2x + a_1$$

Write T as a matrix.

$$\begin{matrix} & 1 & x & x^2 & x^3 \end{matrix}$$

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = T^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad T^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider a problem. A three dimensional object is rotating around the axis pointing in the direction $\vec{\omega}$ at $\|\vec{\omega}\|$ radians per second. if $\vec{r}(t) = (x(t),y(t),z(t))$ is where a point in the object is located and $\vec{v}(t) = \frac{d}{dt}(x(t),y(t),z(t)) = (x'(t),y'(t),z'(t))$ is the velocity of the point then

$$\vec{v}(t) = \vec{\omega} \times \vec{r}(t)$$

or in matrix notation

$$\vec{v} = \frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = A\,\vec{x}(t)$$

Suppose you want to understand an electrical circuit and $\vec{x}(t)$ is a vector describing the voltage differences and current flowing between point in the circuit. Then

$$\frac{d\vec{x}}{dt} = A\vec{x}(t).$$

Again suppose you want to make a stress analysis of a building in a 100 mile per hour wind. The equations boil down to

$$\frac{d\vec{x}}{dt} = A\vec{x}(t) + \vec{w}(t).$$

The point I want to make is that we often want to solve the equation

$$\frac{d\vec{x}}{dt} = A\vec{x}(t)$$

Where A is a (n x n) matrix that does not depend on time. What is the solution? The solution is

$$\vec{x}(t) = e^{tA}\,\vec{x}(0)$$

recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdot$

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\cdot$$

also $e^x = \lim_{n\to\infty} (1+x/n)^n$

$$e^{tA} = \lim_{n\to\infty} (I + \frac{1}{n} A)^n$$

What we are going to learn is how to compute e^{tA}.

Case of diagonal matrix. The product of two diagonal matrices is a diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad B = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \mu_n \end{bmatrix}$$

$$AB = BA = \begin{bmatrix} \lambda_1\mu_1 & 0 & \cdots & 0 \\ 0 & \lambda_2\mu_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n\mu_n \end{bmatrix}$$

So if A is diagonal

$$e^{tA} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{t\lambda_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{t\lambda_n} \end{bmatrix}$$

Eigenvalues and eigenvectors

Suppose A is an (n x n)-matrix. We say λ is an eigenvalue of A if there is a non zero vector $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \lambda\vec{x}$.
Suppose A is a (3 x 3)-matrix and it has three eigenvalue λ_1, λ_2, λ_3 and three eigenvectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$. Let us suppose the eigenvectors are linearly independent. Then each vector $\vec{w}$ is a linear combination of the $\vec{v}$'s. In fact

$$\vec{w} = s_1\vec{v_1} + s_2\vec{v_2} + s_3\vec{v_3}$$

$$A^n\vec{w} = s_1\lambda_1^n\vec{v_1} + s_2\lambda_2^n\vec{v_2} + s_3\lambda_3^n\vec{v_3}$$

So in terms of the $\vec{v}$'s the matrix A is diagonal so

$$e^{tA} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{bmatrix} \qquad \text{in terms of the } \vec{v}$$

Now the traditional basis for $\mathbb{R}^3$ is

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and to compute e^{tA} we have the basis $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$. We form the change of basis matrix S. Given a vector

$$\vec{w} = s_1\vec{v_1} + s_2\vec{v_2} + s_3\vec{v_3}$$

to express $\vec{w}$ in terms of the standard basis we use the matrix