Last time.

A basis for a vector space is a linearly independent set of vectors $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ so that every vector $\vec{w}$ is a linear combination of the basis vectors, i.e.

$$\vec{w} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n}.$$

Theorem. If V is a vector space and $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ is a basis for V and $\{\vec{u_1}, \vec{u_2}, \cdots, \vec{u_m}\}$ is a basis for V then n = m.

Definition. The number of vectors in a basis is called the dimension of the vector space.

Note every real vector space of dimension n is isomorphic to $\mathbb{R}^n$. If $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ is a basis for V and $\vec{w} \in V$ then

$$\vec{w} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n}$$

and the s's are unique so we can represent $\vec{w}$ by $[s_1, s_2, \cdots, s_n]$.

Given an m x n matrix A. There are two subspaces of interest.

$$A \quad \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Null space of A set of vectors in initial space so $A\vec{x} = \vec{0}$.
Range A is the set of vectors $A\vec{x}$ in the image space.

Nullity of A = nullity(A) = dimension of null space of A

Rank of A = rank(A) = dimension of the range of A.

Rank of A is the number of rows of A after row reduction.
Actually, the rank of A is the number of columns of A after column reduction = rank A^T (the rank of the transpose of A)

$$\text{rank}(A) = \text{rank}(A^T).$$

If A is an (m x n)-matrix then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Change of basis.

$B = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ $\qquad$ $C = \{\vec{w_1}, \vec{w_2}, \ldots, \vec{w_n}\}$

$$\vec{v} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n}.$$

$$\vec{v} = r_1\vec{w_1} + r_2\vec{w_2} + \cdots + r_n\vec{w_n}.$$

Suppose you know the s's and you want to calculate the r's

$$P_{C\leftarrow B} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \\ s_n \end{bmatrix}$$

$$P_{C\leftarrow B} = \begin{bmatrix} (\vec{v_1})_1 & (\vec{v_2})_1 & \cdots & (\vec{v_n})_1 \\ (\vec{v_1})_2 & (\vec{v_2})_2 & \cdots & (\vec{v_n})_2 \\ \cdots & \cdots & \cdots & \cdots \\ (\vec{v_1})_n & (\vec{v_2})_n & \cdots & (\vec{v_n})_n \end{bmatrix}$$

with respect to the $\vec{w}$'s

Note $P_{B\leftarrow C} = (P_{C\leftarrow B})^{-1}$

#3 Spring 16

C $\quad \vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

B $\quad$ express $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as $s_1\vec{a} + s_2\vec{b} + s_3\vec{c}$

Compute $P_{C \leftarrow B}$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \vec{a} - 2\vec{b} + \tfrac{1}{2}\vec{c} \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} = \vec{b} - \vec{c} \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} = \tfrac{1}{2}\vec{c}$$

$$P_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \tfrac{1}{2} & -1 & \tfrac{1}{2} \end{bmatrix}$$

$$P_{B \leftarrow C} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 2 \end{bmatrix} \qquad \det(P_{B \leftarrow C}) = 2$$

$$P_{C \leftarrow B} = \tfrac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \tfrac{1}{2} & -1 & \tfrac{1}{2} \end{bmatrix}$$

#8 Spring 16

Let W be the set of (2 x 2) matrices so that $a_{21} = 2a_{12}$.

e.g. $A = \begin{bmatrix} a & b \\ 2b & c \end{bmatrix}$

a) Show $C = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \right\}$ is a basis for W.

b) Find the change of basis matrix $P_{C \leftarrow B}$ where

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$$

Answer to a) is b).

$$P_{B \leftarrow C} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\det(P_{B\leftarrow C}) = 3 + 1 + 0 - (0 + 3 + 0) = 1$$

So $P_{C\leftarrow B} = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ $\quad P_{B\leftarrow C} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$