Determinants. Transpose. $A \leftrightarrow a_{ij}$

$A^T \leftrightarrow a_{ji}$

$\text{Det}(A) = \text{Det}(A^T)$

Row operation $\leftrightarrow$ Column operations

1. Exchange two columns multiplies determinant by -1
2. Multiply a column by s multiplies determinant by s.
3. Replace a column by itself plus a multiple of another column leaves the determinant unchanged.

Cofactor C_{ij} = det (A with i^{th} row and j^{th} column crossed off).

Signed cofactor = $(-1)^{i+j}C_{ij}$

$$(A^{-1})_{ij} = (-1)^{i+j}C^T_{ij}/\text{Det}(A) = (-1)^{i+j}C_{ji}/\text{Det}(A)$$

$$\text{Det}(A) = \sum_{k=1}^{n} a_{ki}(-1)^{i+j}C_{ki} = \sum_{k=1}^{n} a_{ik}(-1)^{i+j}C_{ik}$$

e.g. 3 x 3 matrix.

$$\text{Det}(A) = a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

Last time. A vector space V

Form linear combinations. $s\vec{u} + \vec{v}$

zero vector $\vec{0}$ $\quad \vec{u} + \vec{0} = \vec{u} \quad \vec{u} - \vec{u} = \vec{0} \quad 0\,\vec{u} = \vec{0}$

If $\{\vec{v_1},\vec{v_2},\cdots,\vec{v_n}\}$ is a set of vectors we say $\vec{u}$ is a linear combination of these vectors if $\vec{u} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n}$

A subspace S of a vector space V is a subset of V which is also a vector space (i.e. if $\vec{u}, \vec{v} \in S$ then $a\vec{u} + b\vec{v} \in S$ for all real numbers a and b. Note the set S consisting of one point $\vec{0}$ is a subspace of V. Note V itself is a subspace of V.

If $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\} \in V$ (a vector space) we denote by the Span $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ the set of vectors which are linear combintations of the $\vec{v}$'s. Note span of a set of vectors is a linear subspace.

$$\vec{u} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n}$$

$$\vec{w} = r_1\vec{v_1} + r_2\vec{v_2} + \cdots + r_n\vec{v_n}$$

$$a\vec{u} + b\vec{v} = (as_1+br_1)\vec{v_1} + (as_2+br_2)\vec{v_2} + \cdots + (as_n+br_n)\vec{v_n}$$

A set of vectors $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ are linearly independent if

$$s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n} = 0 \text{ then } s_1 = s_2 = \cdots = s_n = 0.$$

A set of vectors $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ are linearly independent if each vector $\vec{v_j}$ is not a linear combination of the remaining $\vec{v_k}$.

A basis for a vector space V is a set linearly independent set of vectors $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ so that the span of these vectors is all of V (i.e. every vector $\vec{u} \in V$ is a linear combination of the $\vec{v}$'s i.e. $\vec{u} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n}$.

A basis for a vector space is a maximal linearly independent set of vectors $\{\vec{v_1},\vec{v_2},\cdots,\vec{v_n}\}$ in that you can not add a vector to the set and have a linearly independent set of vectors. If you add a vector $\vec{u}$ to the set then

$$-1\vec{u} + s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n} = \vec{0}$$

Theorem. If V is a vector space and $\{\vec{v_1},\vec{v_2},\cdots,\vec{v_n}\}$ is a basis for V and $\{\vec{u_1},\vec{u_2},\cdots,\vec{u_m}\}$ is a basis for V then n = m.

Proof. I remember Professor Towne at Amherst explaining this. The proof is complicated and to understand it you have to sit down and go through it for yourself. I will just give you the ideas.

Start of with

$$\text{Then } \vec{u_1} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_n\vec{v_n}$$

Note all the s's are zero. Let s_k be the one of the non zero s's.

$$\vec{v_k} = s_k^{-1}(\vec{u_1} - s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_{k-1}\vec{v_{k-1}} + s_{k+1}\vec{v_{k+1}} + \cdots + s_n\vec{v_n})$$

So

$$\{\vec{v_1},\vec{v_2},\cdot\cdot,\vec{v_{k-1}},\vec{v_{k+1}},\ \cdot\cdot,\vec{v_n}\ \vec{u_1}\} \text{ is a basis.}$$

Now add $\vec{u_2}$ and cross off another $\vec{v}$

Keep going. Until you run out of $\vec{v}$'s

You can't run out of $\vec{u}$'s because if you did the $\vec{u}$'s would not have been a basis. Hence, $n \leq m$. But repeat the proof with stating with the $\vec{u}$'s and cross off $\vec{v}$'s. You conclude $m \leq n$. Hence, n = m.

How to compute the dimension of a subspace span $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$

Write the vectors as rows

$$\begin{matrix} v_1 & v_2 & & v_n \\ u_1 & u_2 & & u_n \\ w_1 & w_2 & & w_n \end{matrix}$$

row reduce.

Number of rows after row reduction.

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{matrix} \Longrightarrow \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{matrix} \Longrightarrow \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix}$$

Want a basis for the subspace. Take the rows after row reduction.