Last time. How to find an inverse of A

1. Form A¦I
2. Use row operations to get to I¦B

   Then $B = A^{-1}$

If in row reducing you get a row of all zero to the right if ¦ then there is no inverse.

Problem A is a (2 x 2) matrix and

$$A \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & k \end{bmatrix}$$

Find k.

$$B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \qquad AB = C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\det(B) = 1 \quad B^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$A = CB^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{so } k = 1.$$

Determinant.

$$\det(A) = \sum_{\text{all permutations } \pi} \operatorname{sgn}(\pi)\, a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$$

$\pi$ is a permutation of $1,2,\cdots,n$

purmutation of 1,2,3,4

| | |
|---|---|
| 1234 | +1 |
| 1243 | -1 |
| 1324 | -1 |
| 1342 | +1 |
| 1423 | |
| 1432 | |
| 2134 | |
| 2143 | |
| 2314 | |
| 2341 | |

Row reduction

1. exchange row (multiplies determinant by -1)
2. multiply row by constant c (multiplies determinant by c)
3. replace row by itself plus muliple of another row (determinate unchanged)

Upper triangular matrix

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & & a_{3n} \\ 0 & 0 & 0 & \cdots\cdots & a_{nn} \end{matrix}$$

$$\det(A) = a_{11}\, a_{22}\, a_{33} \cdots a_{nn}$$

How to compute the determinant of A.

Start with A   det D = 1

Row reduce A keeping track of D.

1. exchange rows    Multiply D by -1.
2. multiply row by constant c    Divide D by c.
3. replace row by itself plus muliple of another row
   Leave D alone.

If you loose a row then D = 0.

If you do not loose a row then you have 1's on the diagonal and 0's below the diagonal. Then D = determinant.

An example. Find the determinant of A.

$$A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 2 \end{bmatrix} \qquad D = 1$$

Divide row 1 by 2

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 2 \end{bmatrix} \qquad D = 2$$

Replace row 4 by itself minus 4 times row 1.

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \qquad D = 2$$

Divide row 4 by -6.

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = -12$$

Chapter 4.

Linear vector space. $R^n$ vectors $[x_1, x_2, \cdots, x_n]$

Abstract.

Vector space V (over the real numbers)

vectors $\vec{u}, \vec{v}\ \vec{w}$ scalars (real numbers) r,s

Add vectors $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

associative $\vec{u} + (\vec{v} + \vec{w}) = (\vec{v} + \vec{u}) + \vec{w}$

vector $\vec{0}$ $\vec{u} + \vec{0} = \vec{u}$

additive inverse $-\vec{u}$ $\vec{u} + (-\vec{u}) = \vec{0}$

multiply by scalar (real numbers)

$s\vec{u}$

$1\vec{u} = \vec{u}$

$(rs)\vec{u} = r(s\vec{u})$

$s(\vec{u} + \vec{v}) = s\vec{u} + s\vec{v}$

$(r + s)\vec{u} = r\vec{u} + s\vec{u}$

Linear combination $\vec{w}$ is a linear combination of $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n}$ if $w = s_1\vec{u_1} + s_2\vec{u_2} + \cdots + s_n\vec{u_n}$.

e.g. (1,1,1) is a linear combition of (1,1,0) (1,0,1) (0,1,1)

$$(1,1,1) = \tfrac{1}{2}(1,1,0) + \tfrac{1}{2}(1,0,1) + \tfrac{1}{2}(0,1,1)$$

**Definition.** A subspace S of a vector space V is a set of vectors in V so that if $\vec{u}, \vec{v} \in S$ then $s\vec{u} + \vec{v} \in S$.

Note if $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n} \in S$ then $w = s_1\vec{u_1} + s_2\vec{u_2} + \cdots + s_n\vec{u_n} \in S$.

explain why.

$\vec{u_i} \in S$ so $s_1\vec{u_1}, s_2\vec{u_2}, \cdots, s_n\vec{u_n} \in S$

so $s_1\vec{u_1} + s_2\vec{u_2} \in S$

so $s_1\vec{u_1} + s_2\vec{u_2} + s_3\vec{u_3} \in S$

so $s_1\vec{u_1} + s_2\vec{u_2} + \cdots + s_n\vec{u_n} \in S$.

Examples

$R^3$ (all vector (x,y,z) with z = 0 all vectors of the form (x,y,0)

All vectors (x,y,z) so that 2x + y - z = 0

All vectors of the form (s,-7s,11s) s real

If $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n}$ is a set of vectors we denote by the Span($\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n}$) the set of linear combinations $w = s_1\vec{u_1} + s_2\vec{u_2} + \cdots + s_n\vec{u_n}$. Note the span of $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n}$ is a linear subspace S. Why? Becuase a linear combination of linear combinations is again a linear combination e.g.

$$2(3\vec{u_1}+3\vec{u_2}) + 4(-\vec{u_1}+ \vec{u_3} - 5\vec{u_4}) = 2\vec{u_1} + 6\vec{u_2} + 2\vec{u_3} - 20\vec{u_4}$$

Linearly independent vectors

The vectors $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n}$ are linearly independent if no vector $\vec{u_i}$ is a linear combition of the other vectors.

The vectors $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n}$ are linearly independent if

$$s_1\vec{u_1} + s_2\vec{u_2} + \cdots + s_n\vec{u_n} = \vec{0}$$

implies all the $s_i$ are zero (i.e., $s_1 = s_2 = \cdots = s_n = 0$)

The vectors $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n}$ are linearly dependent if they are not linearly independent, if one vector is a linear combination of the other vectors.