Last time

Row reduction.

Want to solve $A\vec{x} = \vec{y}$ Form the augmented matrix $A|Y$

Row reduction.

If you get a row with all zeros, discard it.

If you get a row with all zeros except the last column. No solution.

If the number of rows equals the number of unknowns Solution unique.

If the number of rows equals the number of unknowns minus p then there are p free parameters in the solutions.

The rank of a matrix.

Definition. If A is an m x n matrix then the rank of A is the number of non zero rows after row reduction.

Inverse. Recall the unit or identity matrix I_n is an (n x n)-matrix with 1's down the diagonal and 0's off the diagonal.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A\vec{x} = \vec{x}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{identity matrix unit matrix}$$

If A is an (n x n)-matrix we say A in invertable if there is a matrix B so that AB = BA = I. We denote the inverse of A by A^{-1}. If A is an (n x n)-matrix and A has a left inverse B so BA = I then B is also the right inverse of A so AB = I. (In infinte dimensional space this is not true).

Theorem. Suppose A is an (n x n)-matrix. Then the following statements are equivalent.

(i) Rank(A) = n. Recall the rank of A is the number of rows of A after row reduction.

(ii) The null space of A (the set of vectors $\vec{x}$ so that $A\vec{x} = 0$) is the vector $\vec{0}$ (i.e. if $A\vec{x} = 0$ the $\vec{x} = 0$)

(iii) The range of A is $\mathbb{R}^n$ (i.e. for all $\vec{y} \in \mathbb{R}^n$ there is a vector $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = \vec{y}$)

(iv) The determinate of A is not zero.

Suppose you have to solve the equations

$A\vec{x} = \vec{y}$ for say 50 different vectors $\vec{y}$.

If you know the inverse of A then you can solve directly by multiplying both side of the above equation by A^{-1} and you get

$$A^{-1}A\vec{x} = A^{-1}\vec{y}$$

and since $A^{-1}A = I$ and $I\vec{x} = \vec{x}$ we have

$$\vec{x} = A^{-1}\vec{y}$$

So rather than solve the equation $A\vec{x} = \vec{y}$ fifty times find the inverse A^{-1} and then $\vec{x} = A^{-1}\vec{y}$.

First lets do the two by two case. A two by two matrix has an inverse if and only if it determinate is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

check.

$$AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse of a 3 x 3 matrix.

Determinate of a 3 x 3 matrix

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}$$

Formula for the inverse of a 3 x 3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} a_{22}a_{33}-a_{23}a_{32} & -a_{12}a_{33}+a_{13}a_{32} & a_{12}a_{23}-a_{13}a_{22} \\ -a_{21}a_{33}+a_{23}a_{31} & a_{11}a_{33}-a_{13}a_{31} & -a_{11}a_{23}+a_{13}a_{21} \\ a_{21}a_{32}-a_{22}a_{31} & -a_{11}a_{32}+a_{31}a_{12} & a_{11}a_{22}-a_{12}a_{21} \end{bmatrix}$$

You should be able to compute the inverse of a 3 x 3 matrix in five minutes.

Best method for n x n for $n \geqq 3$. Row reduction.

To find the inverse of A you form the matrix $A \mid I$

e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A \mid I = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Row reduce until you get

$$\left[\begin{array}{ccc|ccc} 1 & a_{12} & a_{13} & x_{11} & x_{12} & x_{13} \\ 0 & 1 & a_{23} & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 1 & x_{31} & x_{32} & x_{33} \end{array}\right]$$

If you get this far there is an inverse.

if you get like below there is no inverse

$$\left[\begin{array}{ccc|ccc} 1 & a_{12} & a_{13} & x_{11} & x_{12} & x_{13} \\ 0 & 1 & a_{23} & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 0 & x_{31} & x_{32} & x_{33} \end{array}\right]$$

Once you get here

$$\left[\begin{array}{ccc|ccc} 1 & a_{12} & a_{13} & x_{11} & x_{12} & x_{13} \\ 0 & 1 & a_{23} & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 1 & x_{31} & x_{32} & x_{33} \end{array}\right]$$

You clear above the bottom diagonal 1 and get here

$$\left[\begin{array}{ccc|ccc} 1 & a_{12} & 0 & x_{11} & x_{12} & x_{13} \\ 0 & 1 & 0 & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 1 & x_{31} & x_{32} & x_{33} \end{array}\right]$$

Then you clear out a_{12} by replacing row 1 by itself minus a_{12} times row 2 and you get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & 1 & 0 & x_{21} & x_{22} & x_{23} \\ 0 & 0 & 1 & x_{31} & x_{32} & x_{33} \end{array}\right]$$

And the matrix of x_{ij} is the inverse

The 3 x 3 case. An example. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & k \end{bmatrix}$$

We form the big matrix [A¦I]

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & k & 0 & 0 & 1 \end{bmatrix}$$

replace row 2 by row 2 - row 1

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 1 & 2 & k & 0 & 0 & 1 \end{bmatrix}$$

replace row 3 by row 3 - row 1

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & k-1 & -1 & 0 & 1 \end{bmatrix}$$

replace row 3 by row 3 - row 2

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & k-2 & 0 & -1 & 1 \end{bmatrix}$$

Now if k=2 there is no inverse.

now you divide row 3 by k-2 the algegra gets messy so assume k = 1 then we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

Now multiply row 3 by -1

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

replace row 1 by itself minus row 2

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

Now replace row 2 by itself minus row 1

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

Inverse of A is A^{-1}

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What kind of question might be on an exam? Here is one.

Find the inverse of the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 2 \end{bmatrix}$$

we make the super augmented matrix

$$\left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

Replace bottom row by itself minus twice top row

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & -1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

Multiply bottom row by -1/3

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1/3 & 0 & 0 & -1/6 \end{array}\right]$$

Replace top row by itself minus twice the bottom row

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1/6 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1/3 & 0 & 0 & -1/6 \end{array}\right]$$

check

$$\begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1/6 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & -1/6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$