Last time

m x n matrix

m rows
n columbs

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix}$$

AX = Y

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

Multiplication of an A (m x n)-matrix times B (n x p) matrix

C = AB

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

If A is an (m x n)-matrix and B is an (n x p) then B is an (m x p) matrix then B is a linear map of a p-dimensional vector space into a n-dimensional vector space and A is a linear map of an n-dimensional vector space into a m-dimensional space. So if we first apply B of a vector in p-space and then apply A to the resulting vector is n-space we get a vector in m-space. Since both A and B are linear maps the resulting map AB (note the matrices are written in reverse order) is a linear map of p-space into m-space which is an m x p matrix C. The formula for C is given by

C = AB

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Unit matrix Identity matrix (n x n)-matrix with 1’s on the diagonal and 0’s every where else

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A\vec{x} = \vec{x}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ identity matrix unit matrix

n x n unit matrix 1’s on the diagonal and 0’s off the diagonal.

$AI = A \qquad IB = B$

for I the identity matrix of appropriate size.

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Solving linear equations

$$AX = Y$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2$$
$$\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_n$$

Augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \cdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_n \end{bmatrix}$$

How to solve (or find there is no solution)

Row operations

1. Exchange two rows
2. Multiply a row by a non zero constant
3. Replace a row by itself plus a multiple of another row.

Convince yourself these operations do not change anything.

i.e. if you have a solution to these equations and apply the row operations you have a solution to the new equations. If you don't have a solution to these equations and you apply any of the row operations you don't have a solution to the new equations.

Leading coefficient of a row is the first non zero entry in the row.

0 0 2 0 1 0 leading coefficient is 2

A matrix is in row echelon form if the rows with all zeros are at the bottom of the matrix. The leading coefficient is one. The leading coefficient of each row is strictly to the right of the leading coefficient in the row above it. example

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 4 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the row operation every matrix can be put in row-echelon form.

First get a one in the a_{11} place. If $a_{11} \neq 0$ divide row by a_{11}. If $a_{11} = 0$ then find a row k where $a_{k1} \neq 0$. Exchange the 1st row and the kth row. Now $a_{11} \neq 0$. Divide first row by a_{11}. Now $a_{11} = 0$. Now for for the second row if $a_{21} \neq 0$ then replace the row by itself minus a_{21} times the first row. After this $a_{21} = 0$. Do the same for the 3d row, then the 4th row until you have all zeros under a_{11}. Now start with the second column starting with a_{22}. Repeat the proceedure so now $a_{22} = 1$ and you have all 0's under a_{22}. Then go to the third column and so on. If at any time you get $a_{nn} = 0$ and all zeros under a_{nn} then just move on to the next column which mean start with a_{nn+1}. When this happens you get the next column starting with one is two over or three over. e.g.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 4 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 & 3 & 4 & 4 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples of row reduction

Simultaneous equations.

$$\begin{aligned} 2y + 6z &= 8 \\ 2x + 2y + 2z &= 6 \\ x - 2y + 2z &= 1 \end{aligned} \qquad \text{augmented matrix} \qquad \left(\begin{array}{ccc|c} 0 & 2 & 6 & 8 \\ 2 & 2 & 2 & 6 \\ 1 & -2 & 2 & 1 \end{array}\right)$$

We want a 1 in the top left corner.

$$\begin{array}{rrr|r} 0 & 2 & 6 & 8 \\ 2 & 2 & 2 & 6 \\ 1 & -2 & 2 & 1 \end{array} \Longrightarrow \text{exchange } (1 \leftrightarrow 3) \quad \begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 2 & 2 & 2 & 6 \\ 0 & 2 & 6 & 8 \end{array}$$

clean up (not necessary but simplifies later arithmetic)

$$\begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 2 & 2 & 2 & 6 \\ 0 & 2 & 6 & 8 \end{array} \Longrightarrow \text{divide rows 2,3 by 2} \quad \begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \end{array}$$

clear out under first 1 in columb 1

$$\begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \end{array} \Longrightarrow \text{row 2} \rightarrow \text{row 2 - row 1} \quad \begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & 3 & 4 \end{array}$$

get a 1 in the (2,2) spot (could divide by 3 but messy)

$$\begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & 3 & 4 \end{array} \Longrightarrow \text{exchange } (2 \leftrightarrow 3) \quad \begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & -1 & 2 \end{array}$$

clear out under 1 in (2,2)

$$\begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & -1 & 2 \end{array} \Longrightarrow \text{row 3} \rightarrow \text{row 3 - 3 x row 2} \quad \begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -10 & -10 \end{array}$$

$$\begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -10 & -10 \end{array} \Rightarrow \text{row 3} \rightarrow \text{-1/10 x row 3} \quad \begin{array}{rrr|r} 1 & -2 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}$$

Once you get hear you can back solve.

solve $z = 1$ $y + 3z = 4$ $y + 3 = 4$ $y = 1$

$x - 2 + 2 = 1$ $x = 1$

answer unique solution $(x,y,z) = (1,1,1)$

Another example

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 2z &= 5 \\ x + 3y + 3z &= k \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 3 & k \end{array}\right) \Longrightarrow \text{row 2 = row 2 - row 1} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 3 & k \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 3 & k \end{array}\right) \Longrightarrow \text{row 3} \rightarrow \text{row 3 - row 1} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & k-3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & k-3 \end{array}\right) \Longrightarrow \text{row 3} \rightarrow \text{row 3 - 2 x row 2} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & k-3-4 \end{array}\right)$$

No solution if $k \neq 7$

if $k = 7$ then $z = t$

$y + t = 2 \quad y = 2 - t$

$x + (2-t) + t = 3 \quad$ so $x = 1$

one parameter family of solutions

$$(x,y,z) = (1,2-t,t)$$

What can happen. Three equations in three unknowns.

Unique solution.

after row reduction

$$\begin{bmatrix} 1 & x & x & c \\ 0 & 1 & x & c \\ 0 & 0 & 1 & c \end{bmatrix}$$

No solution

$$\begin{bmatrix} 1 & x & x & c \\ 0 & 1 & x & c \\ 0 & 0 & 0 & d \end{bmatrix} \quad d \neq 0 \qquad \begin{bmatrix} 1 & x & x & c \\ 0 & 0 & 0 & d \\ x & x & x & x \end{bmatrix} \quad d \neq 0$$

If you get a row will all zeros except the last entry then there is no solution. Example

$$x + y - 3z = 3$$
$$2x + 2y - 6z = 5$$

$$\begin{bmatrix} 1 & 1 & -3 & 3 \\ 2 & 2 & -6 & 5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & -3 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Loose a row

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

last variable $z = t$

second equation $y + z = 2$ so $y = 2 - t$

$$x + 2y + 3z = 3$$
$$x + 2(2-t) + 3t = 3$$
$$x + 4 - t = 3 \quad \text{so} \quad x = t - 1$$

solution is $(t-1, 2-t, t)$