9.14 Stokes' Theorem

- S: a piecewise smooth orientable surface
- C: a piecewise smooth simple closed curve that

bounds S, oriented counter-clockwise as veiwed from above

- $\mathbf{n}$: the unit normal to S, defines orientation of S
- $\mathbf{F}$: $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is a vector field

with $P, Q, R,$ and all first partial derivatives continuous in a region of 3-space containing S

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$$

3 − dim line integral ⇔ surface integral

Use Stokes' Theorem to evaluate $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$

By finding $\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r}}$

Evaluate

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r}} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$$

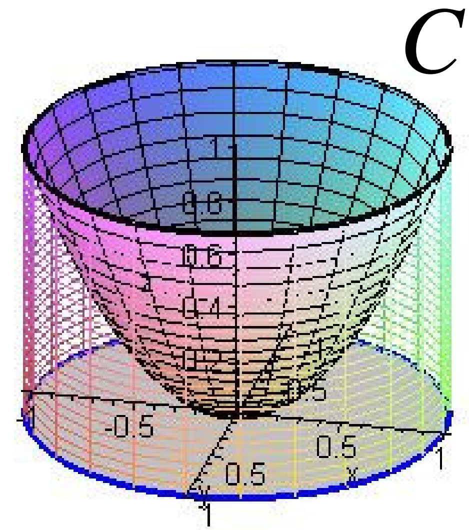
Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$

Evaluate

By finding

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS}$$

Example 1 Use Stokes' Theorem to evaluate $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$ when $\mathbf{F} = \langle y^2 z, xz, x^2 y^2 \rangle$ and S is that part of the paraboloid $z = x^2 + y^2$ that lies in the cylider $x^2 + y^2 = 1,$ oriented upward.



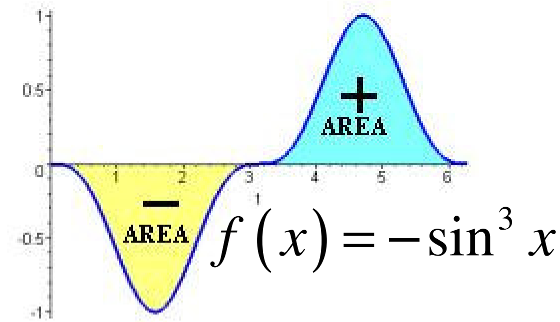
$$\Rightarrow \text{ Find } \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$\text{Parametrize } C: \left.\begin{matrix} x = \cos t \\ y = \sin t \\ z = 1 \end{matrix}\right\} 0 \le t \le 2\pi$$

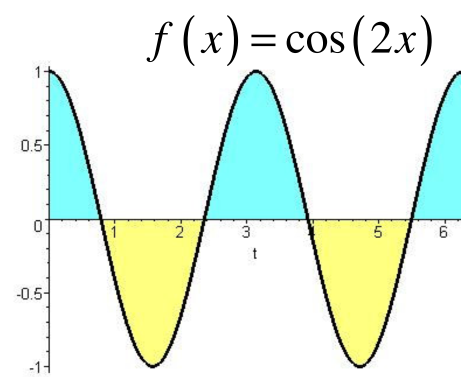
$$\mathbf{r} = \langle \cos t, \sin t, 1 \rangle \Rightarrow d\mathbf{r} = \langle -\sin t, \cos t, 0 \rangle dt$$

$$\mathbf{F} \text{ on } C: \langle \sin^2 t, \cos t, \cos^2 t \sin^2 t \rangle$$

$$\mathbf{F} \cdot d\mathbf{r} = \left(-\sin^3 t + \cos^2 t\right) dt$$



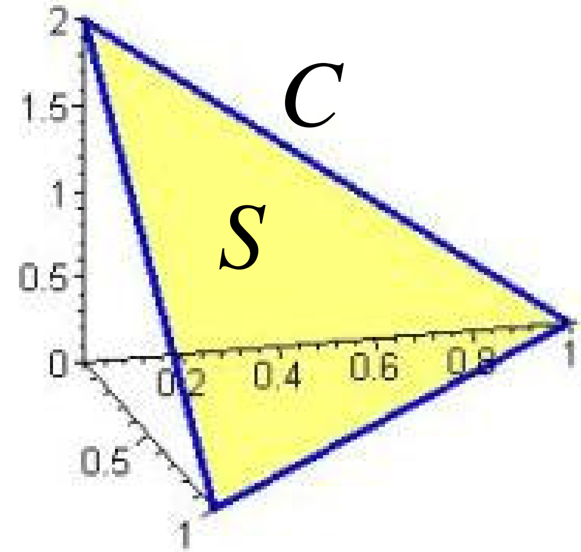
$$\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(-\sin^3 t + \cos^2 t\right) dt$$



$$= \int_0^{2\pi} \cos^2 t dt = \frac{1}{2} \int_0^{2\pi} \left(1 + \cos(2t)\right) dt = \boxed{\pi}$$

Example 2 Use Stokes' Theorem to evaluate $\Rightarrow$ Find $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$

$\oint_C \mathbf{F} \cdot d\mathbf{r}$ when $\mathbf{F} = \langle z^2, y^2, xy \rangle$ and C is the triangle defined by $(1,0,0), (0,1,0)$, and $(0,0,2)$.



S: plane, we need to find the equation using a point and the normal vector to the plane

We can get the normal vector by taking the cross product of two vectors in the plane.

Vector from $(1,0,0)$ to $(0,1,0)$

$v_1 = \langle 0-1, 1-0, 0-0 \rangle = \langle -1, 1, 0 \rangle$

Vector from $(1,0,0)$ to $(0,0,2)$

$v_2 = \langle 0-1, 0-0, 2-0 \rangle = \langle -1, 0, 2 \rangle$

$$\mathbf{v} = v_1 \times v_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \langle 2, 2, 1 \rangle$$

For a plane we could have used $\mathbf{v}$ and made it a unit vector to get $\mathbf{n}$

$S: 2x + 2y + z = d$ plug in any point to find d

$(1,0,0) \rightarrow 2(1) + 2(0) + (0) = d = 2$

$\Rightarrow 2x + 2y + z = 2$ so $\boxed{S: z = 2 - 2x - 2y}$

$g = 2x + 2y + z - 2 = 0$

$\nabla g = \langle 2, 2, 1 \rangle \quad \|\nabla g\| = \sqrt{4+4+1} = 3$

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{1}{3}\langle 2, 2, 1 \rangle$$

$$dS = \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA$$

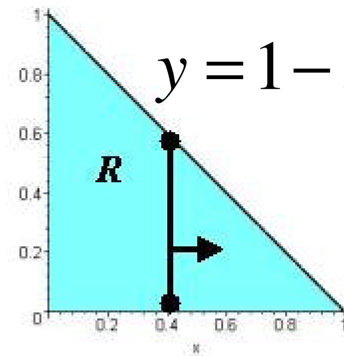
$$dS = \sqrt{1 + (-2)^2 + (-2)^2}\, dA$$

$$dS = 3dA$$

Example 2 (cont.)

$$\mathbf{F} = \left\langle z^2, y^2, xy \right\rangle \qquad \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xy \end{vmatrix} = \left\langle x, 2z - y, 0 \right\rangle$$

$$(\text{curl } \mathbf{F}) = \left\langle x, 2z - y, 0 \right\rangle \qquad \mathbf{n} = \frac{1}{3}\langle 2, 2, 1 \rangle \qquad dS = 3dA$$



$$y = 1 - x$$

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (2x + 4z - 2y)\, dA$$

$$= \int_0^1 \int_0^{1-x} \left[2x + 4(2 - 2x - 2y) - 2y \right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} [8 - 6x - 10y]\, dy\,dx = \int_0^1 \left[(8 - 6x)\, y - 5y^2 \right]_0^{1-x} dx$$

$$= \int_0^1 \left[(8 - 6x)(1 - x) - 5(1 - x)^2 \right] dx = \int_0^1 \left(x^2 - 4x + 3 \right) dx$$

$$= \left(\tfrac{x^3}{3} - 2x^2 + 3x \right)_0^1 = \tfrac{1}{3} - 2 + 3 = \boxed{\frac{4}{3}}$$