8.9 Powers of a Matrix

The Cayley-Hamilton Theorem:

An $n \times n$ matrix A satisfies its characteristic equation.

example:

$A = \begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix}$

Characteristic Eq. :

$\lambda^2 - (\text{trace}(A))\lambda + \det(A) = 0$

$\boxed{\lambda^2 - 5\lambda + 6 = 0}$

by Cayley-Hamilton :

$A^2 - 5A + 6I = 0$

$$A^2 = \begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix} = \begin{pmatrix} 24 & -10 \\ 30 & -11 \end{pmatrix}$$

$$-5A = -5\begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix} = \begin{pmatrix} -30 & 10 \\ -30 & 5 \end{pmatrix}$$

$$6I = 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$A^2 - 5A + 6I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Find a specified power of a matrix A :

Method 1: Using C-H Theorem and repeated multiplication

example:

$A = \begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix}$

Find A^6.

$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow A^2 - 5A + 6I = 0$

Solve for A^2. $\quad \underline{A^2 = 5A - 6I}$

Multiply by A. $\quad A^3 = 5A^2 - 6A$

Plug in for A^2. $\quad A^3 = 5(5A - 6I) - 6A \Rightarrow \underline{A^3 = 19A - 30I}$

Multiply by A. $\quad A^4 = 19A^2 - 30A$

Plug in for A^2. $\quad A^4 = 19(5A - 6I) - 30A \Rightarrow \underline{A^4 = 65A - 114I}$

Multiply by A. $\quad A^5 = 65A^2 - 114A$

Plug in for A^2. $\quad A^5 = 65(5A - 6I) - 114A \Rightarrow \underline{A^5 = 211A - 390I}$

Multiply by A. $\quad A^6 = 211A^2 - 390A$

Plug in for A^2. $\quad A^6 = 211(5A - 6I) - 390A \Rightarrow \underline{A^6 = 665A - 1266I}$

$$A^6 = 665\begin{pmatrix} 6 & -2 \\ 6 & -1 \end{pmatrix} - 1266\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2724 & -1330 \\ 3990 & -1931 \end{pmatrix}$$

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Find a specified power of a matrix A :

Method 2: Using C-H Theorem and a system of equations

example:

$A=\begin{pmatrix}6 & -2\\ 6 & -1\end{pmatrix}$ $\lambda^2-5\lambda+6=0 \Rightarrow A^2-5A+6I=0$

Since $A^2=5A-6I$, every multiple of A will be

Find A^6. $A^m=k_1A+k_0I$, where k_0 and k_1 are real numbers.

As we saw in the previous slide, $m=6 \Rightarrow k_1=665$ and $k_0=-1266$

Goal: To set up a system of equations and solve for k_1 and k_0.

$A^m=k_1A+k_0I$ is a result of the char. eq. so C-H in reverse says each λ satisfies $\lambda^m=k_1\lambda+k_0$. For this A, $\lambda_1=3$ and $\lambda_2=2$, and we want $m=6$.

$$\Rightarrow \begin{matrix}3^6 & = & 3k_1 & + & k_0\\ 2^6 & = & 2k_1 & + & k_0\end{matrix} \Rightarrow \left(\begin{array}{cc|c}3 & 1 & 729\\ 2 & 1 & 64\end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{cc|c}1 & 0 & 665\\ 2 & 1 & 64\end{array}\right)$$

$$\xrightarrow{-2R_1+R_2} \left(\begin{array}{cc|c}1 & 0 & 665\\ 0 & 1 & -1266\end{array}\right) \Rightarrow \boxed{k_1=665 \text{ and } k_0=-1266}$$

Find the inverse of a matrix using the Cayley-Hamilton Theorem

example:

$\begin{pmatrix}2 & 0 & 1\\ -2 & 3 & 4\\ -5 & 5 & 6\end{pmatrix}$

$$\lambda^3-\left(trace(A)\right)\lambda^2+\underbrace{\left(C_{11}+C_{22}+C_{33}\right)}_{\text{sum of the diagonal cofactors}}\lambda-\det(A)=0$$

$$\lambda^3-11\lambda^2+(-2+17+6)\lambda-1=0 \Rightarrow \underline{\lambda^3-11\lambda^2+21\lambda-1=0}$$

C-H $\Rightarrow A^3-11A^2+21A-I=0$

Solve for I. $I=A^3-11A^2+21A$

Multiply by A^{-1}. $A^{-1}=A^2-11A+21I$

$$A^{-1}=\begin{pmatrix}-1 & 5 & 8\\ -30 & 29 & 34\\ -50 & 45 & 51\end{pmatrix}-11\begin{pmatrix}2 & 0 & 1\\ -2 & 3 & 4\\ -5 & 5 & 6\end{pmatrix}+21\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A^{-1}=\begin{pmatrix}-2 & 5 & -3\\ -8 & 17 & -10\\ 5 & -10 & 6\end{pmatrix}$$

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