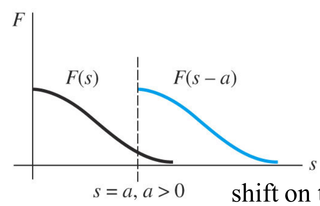
Section 4.3
Translation Theorems

Laplace Transform $L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt = F(s)$

$$L\{e^{at} f(t)\} = \int_0^{\infty} e^{at} f(t) e^{-st}\,dt = \int_0^{\infty} f(t) e^{-(s-a)t}\,dt = F(s-a)$$

a any real number

First Translation Theorem



shift on the s − axis

laplace of $f(t)$ with $s-a$ replacing s

We've seen this translation theorem in action already when we derived both

$$L\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{for integer } n > 0, \quad s > 0$$

$$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \quad \text{for integer } n > 0, \quad s > a$$

We derived

$$L\{\sin(bt)\} = \frac{b}{s^2 + b^2}, \; s > 0$$

Now, instead of using the definition, we can use the translation theorem to find

$$L\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2 + b^2}, \; s > a$$

Working backwards, we have:

$$L^{-1}\{F(s-a)\} = e^{at} f(t)$$

example 1:

$$L^{-1}\left\{\frac{1}{s^2-6s+13}\right\}$$

We need to complete the square in the denominator.

$$L^{-1}\left\{\frac{1}{s^2-6s+\underline{\ 9\ }+13-\underline{\ 9\ }}\right\} = L^{-1}\left\{\frac{1}{(s-3)^2+4}\right\} = L^{-1}\left\{\frac{1}{2}\frac{2}{(s-3)^2+4}\right\} = \boxed{\tfrac{1}{2}e^{3t}\sin 2t}$$

$(\frac{1}{2} \text{ of } 6)^2$

example 2:

$L\{\cos(bt)\} = \frac{s}{s^2+b^2}$, $s>0$

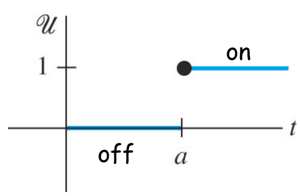
$$L^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = L^{-1}\left\{\frac{s}{s^2+4s+\underline{\ 4\ }+5-\underline{\ 4\ }}\right\} = L^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}$$

need $s+2$ in the numerator

$$= L^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1}\right\} = \boxed{e^{-2t}\cos t - 2e^{-2t}\sin t}$$

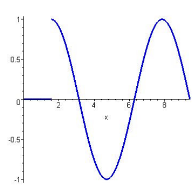
Unit Step Function

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$



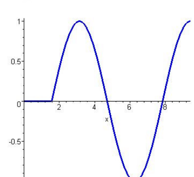
$$f(t)\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t), & t \ge a \end{cases} = \begin{cases} f(t)\,\textbf{OFF}, & 0 \le t < a \\ f(t)\,\textbf{ON}, & t \ge a \end{cases}$$

$$\sin t\,\mathcal{U}(t-\tfrac{\pi}{2}) = \begin{cases} 0, & 0 \le t < \frac{\pi}{2} \\ \sin t, & t \ge \frac{\pi}{2} \end{cases}$$



0 up to $\frac{\pi}{2}$ and then picks up $\sin t$ at $\frac{\pi}{2}$

$$\sin(t-\tfrac{\pi}{2})\mathcal{U}(t-\tfrac{\pi}{2}) = \begin{cases} 0, & 0 \le t < \frac{\pi}{2} \\ \sin(t-\frac{\pi}{2}), & t \ge \frac{\pi}{2} \end{cases}$$



0 up to $\frac{\pi}{2}$ and then shift $\sin t$ and start it at $\frac{\pi}{2}$

shift on the t – axis

Second Translation Theorem

$$L\{f(t-a)U(t-a)\}=e^{-as}L\{f(t)\}$$

a any real number

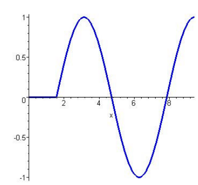
$$L\{f(t-a)U(t-a)\}=\int_0^\infty f(t-a)U(t-a)e^{-st}dt$$

$$=\int_0^a f(t-a)\underbrace{U(t-a)}_{0 \text{ for } 0\le t<a}e^{-st}dt+\int_a^\infty f(t-a)\underbrace{U(t-a)}_{1 \text{ for } t\ge a}e^{-st}dt$$

$$=\int_a^\infty f(t-a)e^{-st}dt \qquad \boxed{\begin{array}{ll} w=t-a & t=w+a \\ dw=dt & t=a\Rightarrow w=0\end{array}}$$

$$=\int_0^\infty f(w)e^{-s(w+a)}dw \;=e^{-as}\overbrace{\int_0^\infty f(w)e^{-sw}dw}^{L\{f(t)\}} \;=e^{-as}L\{f(t)\}$$

$$L\{\sin(t-\tfrac{\pi}{2})U(t-\tfrac{\pi}{2})\}=\frac{e^{\frac{-\pi}{2}s}}{s^2+1}$$



$$f(t)=\begin{cases} g(t), & 0\le t<a \\ h(t), & t\ge a\end{cases} \Rightarrow f(t)=g(t)-g(t)U(t-a)+h(t)U(t-a)$$

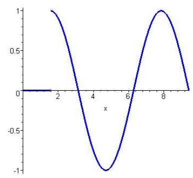
$$f(t)=\begin{cases} 0, & 0\le t<a \\ g(t), & a\le t<b \\ 0, & t\ge b\end{cases} \qquad \Rightarrow f(t)=g(t)\left[U(t-a)-U(t-b)\right]$$

Alternative form of 2nd Translation Theorem

$$L\{f(t)U(t-a)\}=e^{-as}L\{f(t+a)\}$$

a any real number

$$L\left\{\sin(t)U\left(t-\tfrac{\pi}{2}\right)\right\}=e^{\frac{-\pi}{2}s}L\left\{\sin\left(t+\tfrac{\pi}{2}\right)\right\}$$



$$=e^{\frac{-\pi}{2}s}L\left\{\cos(t)\right\}$$

$$=\frac{e^{\frac{-\pi}{2}s}s}{s^2+1}$$