The Impossibilities of Voting

- Introduction
- Majority Criterion ✓ _One with > 50% should be chosen_
- Condorcet Criterion ✓ _Pairwise Comp Winner should be chosen_
- Monotonicity Criterion
- Irrelevant Alternatives Criterion
- Arrow's Impossibility Theorem

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Introduction

Different voting methods applied to the same voter profile can result in different outcomes. The voting methods presented in the previous section are considered fair and reasonable.

Defects in the plurality method, the pairwise comparison method, the Borda method, and the Hare method are revealed by considering the following (next slide) four desirable attributes for any voting method.

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Introduction

1. the *majority criterion*
2. the *Condorcet criterion*
3. the *monotonicity criterion*
4. the *irrelevant alternatives* (IA) *criterion*

The first two concern desirable qualities for a voting method when it is used a single time to determine a winner. The third and fourth criteria concern desirable qualities for a voting method when it is used twice in an election procedure.

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The Majority Criterion

If a candidate has a majority of first-place rankings in a voter profile, then that candidate should be the winner of the election.

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Example: The Borda Method and the Majority Criterion

Members of a club are planning for an upcoming event. The dinner choices include beef (b), chicken (c), or pork (p). The members rank their choices according to the voter profile below. Show that the Borda method does not satisfy the majority criterion.

Number of Voters	Ranking
9	b > p > c
4	c > p > b
4	p > c > b

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Example: The Borda Method and the Majority Criterion

Solution   *Beef Majority*

Number of Voters	Ranking	1st place (2 points)	2nd Place (1 point)	3rd Place (0 points)
9	b > p > c	b = 2	p = 1	c 0
4	c > p > b	c = 2	p = 1	b 0
4	p > c > b	p = 2	c = 1	b

Beef   9(2) = 18

Chicken   4(2) + 4(1) = 12

Pork = 9(1) + 4(1) + 4(2) = 9 + 4 + 8 = 21

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Example: The Borda Method and the Majority Criterion

Solution (continued)

The winner of the Borda method is pork even though beef had a majority of the first-place votes (9). We see that the majority criterion is not satified.

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The Condorcet Criterion

A candidate who can win a pairwise comparison with every other candidate is called a **Condorcet candidate**.

If a Condorcet candidate exists for a voter profile, then the Condorcet candidate should be the winner of the election

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Example: Plurality Method and the Condorcet Criterion

Revisiting the dinner choices of beef (b), chicken (c), or pork (p) from a previous example, the ranked choices according to voter profile are listed below. Show that the plurality method fails to satisfy the Condorcet criterion.

Number of Voters	Ranking
7	b > p > c
6	c > p > b
5	p > c > b

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Example: Plurality Method and the Condorcet Criterion

Solution

From last section, we have the pairwise comparison winner as pork. By the plurality method, beef is the winner. Since the Condorcet candidate was not selected by the plurality method, this method does not satisfy the Condorcet criterion.

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Monotonicity Criterion

If Candidate x wins an election and, before a second election, the voters who rearrange their rankings move Candidate x to the top of their rankings, then Candidate x should win the second election.

Showing That the Hare Method Fails to Satisfy the Monotonicity Criterion

The College Art Association increased its executive board to 29 members. Voting for the site of the yearly meeting is among Miami m, Boston b, Chicago c, and Seattle s. The board members agree to use the Hare method to make their selection. They decide to hold a preliminary, nonbinding vote Friday afternoon and to meet again Saturday morning for the official vote. The Friday afternoon voter profile of the 29 members is given in **Table 15**. Show that the monotonicity criterion is violated in this case.

Table 15

Number of Voters	Ranking
7	m > b > s > c
8	b > c > m > s
10	c > s > m > b
2	m > c > s > b
2	m > s > b > c

Handwritten annotations:

29

majority = 15 or more

Hare

Round 1
1st place votes

7+2+2 = 11 Miami
8 = Boston
10 = Chicago
0 = Seattle
throw out Seattle

Round 2
Since seattle was 0
Same result
11 Miami
8 Boston
10 Chicago
throw out Boston

Round
miami 11
Chicago 18 majority ✓
Boston's 8 votes went to Chicago
Chicago wins

Friday evening, Chicago supporters wine and dine the four voters with the two bottom rankings in **Table 15**. They convince the four voters to rearrange their rankings, placing Chicago first.

Table 16 shows the voter profile for the official vote on Saturday morning. The four voters with the bottom rankings have moved Chicago into first place. The other 25 voters do not rearrange their rankings. Chicago supporters believe the four additional first-place rankings will work in their favor.

Table 16

Number of Voters	Ranking
7	m > b > s > c
8	b > c > m > s
10	c > s > m > b
2	c > m > s > b
2	c > s > b > m

Binding vote
7   [m > b >] s > c
8   b > c > m > s
10   c > s > m > b
2   c > m > s > b
2   c > s > b > m

Round 1		Round 2		Round 3	
Chicago	14	Chicago	14 same	Boston 8+7 = 15 ← from Miami
Miami	7	Miami	7	Chicago 14 2 majority
Boston	8	Boston	8	
Seattle	0	throw out		**Boston Wins**
throw out seattle		Miami		

Irrelevant Alternatives (IA) Criterion

If Candidate *x* wins a first election and one (or more) of the losing alternatives drops out before a second vote, the winner *x* of the first election should win the second election.

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Example: IA and Plurality

For the event described earlier, recall that the dinner choices include beef (b), chicken (c), or pork (p). Under the plurality method beef is chosen as the winner.

Number of Voters	Ranking
7	b > p > c
6	c > p > b
5	p > c > b

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Example: IA and Plurality

If however, pork is removed from the ballot before a second vote, those that voted for pork first, now vote for chicken and chicken wins with 11 first-place votes.

Number of Voters	Ranking
7	b > p > c
6	c > p > b
5	p > c > b

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Example: IA and Plurality

We see that under the plurality method the irrelevant alternatives criterion may not be satisfied.

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Arrow's Impossibility Theorem

For an election with more that two alternatives, there does not exist any voting method that simultaneously satisfies the majority criterion, the Condorcet criterion, the monotonicity criterion, and the irrelevant alternatives criterion.

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Summary of Desirable Criteria and Voting Methods That Satisfy Them

	Plurality Method	Pairwise Comparison	Borda Method	Hare Method
Majority Criterion	satisfied	satisfied	not satisfied	satisfied
Condorcet Criterion	not satisfied	satisfied	not satisfied	not satisfied
Monotonicity Criterion	satisfied	not satisfied	not satisfied	not satisfied
IA Criterion	not satisfied	not satisfied	not satisfied	not satisfied

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